Gravitational-wave Astrophysics
Summer term 2024

Due July 3, 2024 at 10am via email to TA, or on paper during lecture

PROBLEM SET 6

6.1. Neutron star structure [8 pts.]

The TOV equations describe a spherically-symmetric ideal fluid in general-relativistic hydrostatic equilibrium. In this exercise we solve the TOV equations to compute the interior structure of neutron stars. We then try to constrain the equation of state of neutron star matter using the GW170817 binary neutron-star observation by LIGO and Virgo.

We can formulate the TOV equations as two coupled ODEs for the pressure $p(r)$ and interior mass $m(r)$:

$$\frac{dp}{dr} = -\frac{(\rho + p)\left(m + 4\pi r^3 p\right)}{r\,(r - 2m)}, \tag{1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{2}$$

To solve the TOV equations we have to specify an *equation of state (EOS)* that relates pressure p and density ρ. Furthermore, to integrate Eqs. (1) and (2) starting at $r = 0$, we need initial conditions. We set $m(r = 0) = 0$ and choose a central density $\rho(r = 0) = \rho_{\text{central}}$.

(a) First, we choose a simple *polytropic* EOS that approximates the more sophisticated models for neutron stars fairly well around nuclear density $\rho_{\text{nuclear}} \approx 2.3 \times 10^{17} \frac{\text{kg}}{\text{m}^3}$:

$$p(\rho_0) = 3 \times 10^{32} \left(\frac{\rho_0}{\rho_{\text{nuclear}}}\right)^3 \frac{\text{N}}{\text{m}^2} \tag{3}$$

Note that the EOS is given in terms of *rest-mass-density* ρ_0. However, the TOV Equations (1) and (2) require the total energy density ρ. The two are related by

$$\rho = C\rho_0 + \frac{p}{\Gamma - 1} \tag{4}$$

where $\Gamma = 3$ is the exponent in Eq. (3) and here $C = 1$. The difference between ρ and ρ_0 is the internal energy density, in essence, the energy added to the material through the compression up to the current pressure p (and indeed, Eq. (4) can be derived from the first law of thermodynamics).

Run the Python notebook `neutron-star-structure.ipynb`, which is supplied with the problemset, to integrate the TOV equations for this EOS. For a given central density ρ_{central} we obtain a profile $p(r)$, $m(r)$ and $\rho(r)$ for the neutron star (Fig. 1 in the notebook). Briefly describe how we find the neutron-star radius R and its mass M, and how they relate to $p(r)$ and $m(r)$.

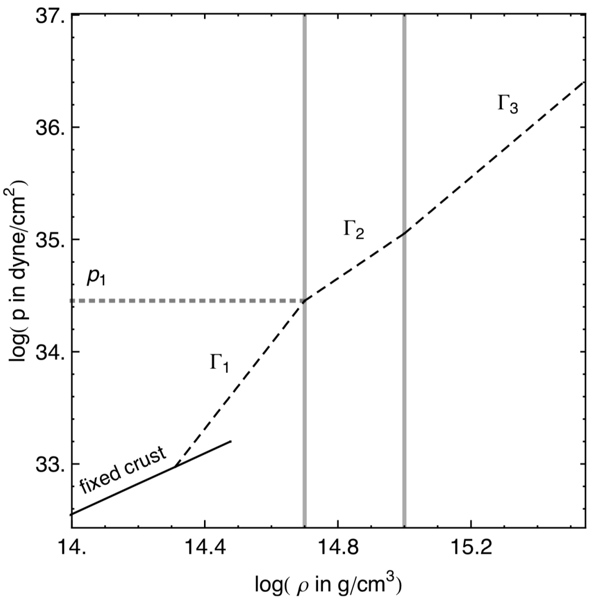
(b) Next, we compute neutron star profiles for a range of central densities. Discuss the results presented in Fig. 2 and 3 in the notebook. *Hint:* configurations where $\frac{dM}{d\rho_{\text{central}}} < 0$ are unstable, i.e. the neutron star would collapse to a black hole.

(c) The extreme matter conditions in the cores of neutron stars are laboratories for nuclear physics. Theoretical models of nuclear interactions make predictions for the EOS of the neutron star matter that we can test with (multimessenger) astrophysical observations. We can approximate many of these EOS models as a series of polytropes

$$p_i(\rho_0) = K_i \rho_0^{\Gamma_i} \tag{5}$$

where each is valid in a particular density interval (see diagram below). Following [Read et al (2018)][1] we begin with the adiabatic index $\Gamma_0 = 1.33$ at sub-nuclear densities that occur in the neutron star crust (taken from [Douchin, Haensel (2001)][2]). We then parametrize three layers of the neutron star core that are divided at densities $\rho_{0,1} = 10^{14.7} \frac{\text{g}}{\text{cm}^3}$ and $\rho_{0,2} = 10^{15} \frac{\text{g}}{\text{cm}^3}$. These two densities are indicated as the grey vertical lines in the figure below. Furthermore, as indicated in the figure, let p_1 be the pressure at density $\rho_{0,1}$ and let the adiabatic indices of the three parts be Γ_1, Γ_2 and Γ_3.

Derive the constants of proportionality K_0, K_1, K_2 and K_3 in terms of these parameters from continuity of p. Derive also the coefficients C_1, C_2 and C_3 from continuity of the total energy density ρ, and from $C_0 = 1$.



(d) The `polytropes` dictionary in the Python notebook lists a few polytropic approximations to EOS models by specifying their p_1 and Γ_i parameters. They are plotted in Fig. 4 in the notebook, along with the simple polytrope Eq. (3). Compute the mass-radius relations for these EOS models (Fig. 5 and 6 in the notebook).

Given that neutron stars with $\sim 2M_\odot$ were already observed (e.g. the millisecond pulsars *PSR J0348+0432* and *PSR J1614-2230*), what can you conclude for the realism of the EOS model *GS1*?

(e) Add a few more equations of state from Table III in [Read et al (2018)] to the `polytropes` dictionary in the notebook and compute their mass-radius relations (Fig. 5 and 6 in the notebook). Fig. 7 also includes results inferred from the first binary neutron-star merger observation GW170817 by LIGO and Virgo (taken from Fig. 3 (left panel) of [LVC (2018)][3]). Which EOS models that you tried are compatible with the observations and which are not?

Bonus: Consult [Read et al (2018)] and its references to find out details about your favourite EOS model and summarize.

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[1] Read et al (2018), Constraints on a phenomenologically parameterized neutron-star equation of state, https://arxiv.org/abs/0812.2163

[2] Douchin and Haensel (2001), A unified equation of state of dense matter and neutron star structure, https://arxiv.org/abs/astro-ph/0111092

[3] LIGO and Virgo collaborations (2018), GW170817: Measurements of Neutron Star Radii and Equation of State, https://dcc.ligo.org/LIGO-P1800115/public

6.2. Gravitational waves from pulsars [8 pts.]

Neutron stars possess a rigid crust that is 10 billion times stronger than steel and can support a "mountain" of up to a few cm height. If a neutron star with such a non-axisymmetric perturbation rotates, it will emit gravitational radiation. The GW emission will in turn slow down the rotation of the neutron star. This exercise explores this process.

We recall some mechanics of rotating bodies: The rotational dynamics of a rigid body are determined by its *inertia tensor* that, in Cartesian coordinates, is

$$J_{ij} = \int d^3x \rho(\boldsymbol{x}) \left(r^2 \delta_{ij} - x^i x^j\right). \tag{6}$$

J_{ij} has three principal axes which co-rotate with the body. In the (rotating) coordinate system aligned with the principal axes, the inertia tensor is diagonal

$$\bar{J}_{ij} = \mathrm{diag}(J_1, J_2, J_3). \tag{7}$$

J_1, J_2 and J_3 are the body's principal moments of inertia.

We will consider a neutron star rotating with angular frequency Ω around its principal axis $\boldsymbol{e}_3$. We further assume that the neutron star has a deformation such that $J_1 \neq J_2$ (for example, a "mountain" on the equator).

(a) Express the inertia tensor (7) in inertial (nonrotating) coordinates such that the z-axis is aligned with the principal axis of rotation $\boldsymbol{e}_3$. *Hint:* construct the rotation matrix $\mathbf{R}_z(\Omega t)$ of a rotation around the z-axis by the angle Ωt. Then compute

$$\mathbf{J} = \mathbf{R}_z(\Omega t)^{\mathrm{T}} \cdot \mathrm{diag}(J_1, J_2, J_3) \cdot \mathbf{R}_z(\Omega t). \tag{8}$$

The resulting $\mathbf{J} = J_{ij}$ is time-dependent and would agree with the result of Eq. (6), had we evaluated the integral for the actual rotating mass-distribution of the neutron star.

(b) Show that the trace-free part of the inertia tensor J_{ij} is equal to the negative of the trace-free part of the quadrupole moment I_{ij} (which we also called the *reduced* quadrupole moment $\mathcal{I}_{ij}$ in class). Using this equality, and your result from part (a), show that the power radiated in gravitational waves by the rotating neutron star is

$$L_{\mathrm{GW}} = \frac{32}{5} \left(J_3 \epsilon\right)^2 \Omega^6 \quad \text{with the } \textit{ellipticity} \quad \epsilon \equiv \frac{J_1 - J_2}{J_3}. \tag{9}$$

Hint: recall from the lecture that

$$L_{\mathrm{GW}} = -\frac{dE}{dt} = \frac{1}{5} \langle \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}^{ij} \rangle. \tag{10}$$

(c) Consider a neutron star that is approximated as a uniform density sphere with mass $1.4 M_\odot$ and radius $R = 10$km, so that $J_3 \sim \frac{2}{5} M R^2 \sim 10^{45} \mathrm{g\,cm}^2$. Assume it has a rotational period of $P = 33$ms, like the Crab pulsar *PSR B0531+21*. Its rotational energy is $E = \frac{1}{2} J_3 \Omega^2$. Find the spin-down rate $\dot{\Omega}$ caused by energy radiated away in gravitational waves. Show that for a fiducial ellipticity of $\epsilon = 10^{-7}$ the rate of change in frequency is small and thus the GWs are approximately monochromatic over an observation time of a few years. *Hint:* don't forget to reinstate factors of G_{N} and c.

(d) The observed spin-down rate of the Crab pulsar is $\dot{P} = 4.2 \times 10^{-13} \frac{s}{s}$. Assuming that the spin-down is solely caused by GW emission, what would the ellipticity of the Crab pulsar need to be to explain this value?

(e) In several pulsars the spin-down rate has been measured with pulsar-timing observations and is generally quantified by a *braking index* n defined by $\dot{\Omega} \propto \Omega^n$. For the Crab pulsar $n \approx 2.5$, while for the Vela pulsar $n \approx 1.5$. For pure electromagnetic dipole radiation we would find $n = 3$. Read off the braking index for GW-dominated spin-down from your results in (c). Is GW emission the dominant mechanism for the spin-down of the Crab pulsar?

6.3. Gravitational waves from merging supermassive black holes [4 pts.]

(Hartle, Ch. 23, Problem 19): Suppose for simplicity that (1) every galaxy contains a $10^9 M_\odot$ black hole, (2) that every galaxy merges once in its lifetime, and (3) that when they do, the black holes in their cores coalesce. Consider a detector built to detect the gravitational waves from such events. Even though they do not really apply, use the results of linearized gravity to:

(a) Estimate the frequency range in which the detector would have to operate.

(b) Estimate the strain sensitivity that would be necessary to see mergers out to the edge of the visible universe.

(c) Estimate the duration of such events in usual time units.

(d) Estimate the rate at which such events would be detected.