Note for hQCD with B

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1 Gravity setup

Action:

$$S+S_{\partial}=\frac{1}{2\kappa_N^2}\int d^5x\sqrt{-g}\left[\mathcal{R}-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-\frac{Z(\phi)}{4}F_{\mu\nu}F^{\mu\nu}-V(\phi)\right]$$
$$+\frac{1}{2\kappa_N^2}\int_{r\to\infty}dx^4\sqrt{-h}\left[2K-6-\frac{1}{2}\phi^2-\frac{6\gamma^4-1}{12}\phi^4\ln[r]-b\phi^4+\frac{(c_0e^{-c_1^2c_2^2}+c_3e^{-c_4^2c_5^2})}{4}F_{\rho\lambda}F^{\rho\lambda}\ln[r]\right],$$

where

$$Z[\phi]=c_0e^{-c_1^2(\phi-c_2)^2}+c_3e^{-c_4^2(\phi-c_5)^2},\quad V[\phi]=\left(6\gamma^2-\frac{3}{2}\right)\phi^2-12\cosh(\gamma\phi)+\mathrm{gm}\,\phi^4+\mathrm{gn}\,\phi^6.$$

Metric:

$$ds^2=-f(r)e^{-\eta(r)}dt^2+\frac{dr^2}{f(r)}+r^2\left[dx^2+dy^2+g(r)dz^2\right],$$

where

$$\phi=\phi(r),\quad A_\mu dx^\mu=A_t(r)dt+\frac{B}{2}(xdy-ydx).$$

By means of the metric ansatz, one can obtain:

$$\sqrt{-g}=e^{-\frac{\eta(r)}{2}}r^3\sqrt{g(r)},\quad F_{\mu\nu}F^{\mu\nu}=\frac{2B^2}{r^4}-2e^{\eta(r)}A_t'(r)^2,\quad \sqrt{-h}=e^{-\frac{\eta(r)}{2}}r^3\sqrt{f(r)}\sqrt{g(r)}$$

The energy-momentum tensor reads:

$$T_{tt}=\frac{1}{48}(12B^2(c_0e^{-c_1^2c_2^2}+c_3e^{-c_4^2c_5^2})+(1+48b)\phi_s^4+48\phi_s\phi_v-144f_v)+\frac{4g_v}{g_0},$$
$$T_{xx}=T_{yy}=P_x=P_y=\frac{1}{48}(-8B^2(c_0e^{-c_1^2c_2^2}+c_3e^{-c_4^2c_5^2})+(3-48b+16gm-8\gamma^4)\phi_s^4+48\phi_s\phi_v-48f_v),$$
$$T_{tt}=P_z=\frac{1}{48}g_0(4B^2(c_0e^{-c_1^2c_2^2}+c_3e^{-c_4^2c_5^2})+(3-48b+16gm-8\gamma^4)\phi_s^4+48\phi_s\phi_v-48f_v)+4g_v.$$

2 UV and IR expansion

Variable substitution :

$$r=\frac{1}{z},\quad \phi(r)=z\Phi(z),\quad f(r)=\frac{F(z)}{z^2},\quad \eta(r)=\Sigma(z),\quad A_t(r)=A(z),\quad g(r)=G(z)$$

UV expansion (See UV exapnsion with B.nb):

$$PhiUV=\phi_s+\left[\phi_v+\frac{1}{6}(1+12gm-6\gamma^4)\phi_s^3\ln(z)\right]z^2+\cdots,$$
$$FUV=1+\frac{\phi_s^2}{6}z^2+\left(f_v+\frac{1}{12}\left(B^2\left(6c_0e^{-c_1^2c_2^2}+6c_3e^{-c_4^2c_5^2}\right)+\left(1+12gm-6\gamma^4\right)\phi_s^4\right)\ln(z)\right)z^4+\cdots,$$
$$\Sigma(z)=\eta_0+\frac{\phi_s^2}{6}z^2+\cdots,$$
$$G(z)=g_0+\left(g_v+\frac{1}{4}B^2\left[c_0e^{-c_1^2c_2^2}+c_3e^{-c_4^2c_5^2}\right)g_0\ln(z)\right]z^4+\cdots.$$

IR expansion (See IR exapnsion with B.nb):

$$\begin{aligned}
\Phi IR &= \phi_h + Phi1(z-1) + Phi2(z-1)^2 + \cdots ,\\
FIR &= F1(z-1) + F2(z-1)^2 + \cdots ,\\
\Sigma IR &= Eta(z-1) + Eta2(z-1)^2 + \cdots ,\\
GIR &= 1 + g1(z-1) + g2(z-1)^2 + \cdots .
\end{aligned}$$

3 Scaling symmetry

1) $t \to \lambda_t t, \quad e^{\eta(r)} \to \lambda_t^2 e^{\eta(r)}, \quad A_t(r) \to \lambda_t^{-1} A_t(r)$
2) $r \to \lambda_r r, \quad (t, \mathbf{x}) \to \lambda_r^{-1}(t, \mathbf{x}), \quad f(r) \to \lambda_r^2 f(r), \quad A_t(r) \to \lambda_r A_t(r), \quad B \to \lambda_r^2 B$
3) $x_3 \to \lambda_3 x_3 \quad g(r) \to \lambda_3^{-2} g(r)$

4 Magnetic susceptibility

In this section, we use "tilde" to represent real physical quantities.

$$\begin{aligned}
\chi_B =& \frac{\partial M}{\partial B}\Big|_{B=0} = -\frac{\partial^2 f}{\partial B^2}\Big|_{B=0} = \frac{\partial^2 (S+S_\partial)}{\partial B^2}\Big|_{B=0}\\
=& -\frac{1}{2\kappa_N^2}\int_{\tilde r_h}^{\infty} \frac{e^{-\frac{\tilde\eta(\tilde r)}{2}}\sqrt{\tilde g(\tilde r)}Z[\tilde\phi(\tilde r)]}{\tilde r}\,d\tilde r + \frac{1}{2\kappa_N^2}\lim_{\tilde r\to\infty}\frac{e^{-\frac{\tilde\eta(\tilde r)}{2}}\sqrt{\tilde f(\tilde r)}\sqrt{\tilde g(\tilde r)}(c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})}{\tilde r}\ln(\tilde r)\\
=& -\frac{1}{2\kappa_N^2}\int_{\tilde r_h}^{\infty} \frac{e^{-\frac{\tilde\eta(\tilde r)}{2}}\sqrt{\tilde g(\tilde r)}Z[\tilde\phi(\tilde r)]}{\tilde r}\,d\tilde r + \frac{1}{2\kappa_N^2}\lim_{\tilde r\to\infty}(c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})\ln(\tilde r)\\
=& -\frac{1}{2\kappa_N^2}\int_{\tilde r_h}^{\infty} \frac{e^{-\frac{\tilde\eta(\tilde r)}{2}}\sqrt{\tilde g(\tilde r)}Z[\tilde\phi(\tilde r)] - (c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})}{\tilde r}\,d\tilde r + \frac{1}{2\kappa_N^2}(c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})\ln(\tilde r_h)\\
=& -\frac{1}{2\kappa_N^2}\int_{\lambda_r r_h}^{\lambda_r\infty} \frac{\lambda_t^{-1} e^{-\frac{\eta(r)}{2}}\lambda_3^{-1}\sqrt{g(r)}Z[\phi(r)] - (c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})}{r}\,dr + \frac{1}{2\kappa_N^2}(c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})\ln(\lambda_r r_h)\\
=& -\frac{1}{2\kappa_N^2}\int_{\lambda_r^{-1}0}^{\lambda_r^{-1} z_h} \frac{\lambda_t^{-1} e^{-\frac{\Sigma(z)}{2}}\lambda_3^{-1}\sqrt{G(z)}Z[z\Phi(z)] - (c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})}{z}\,dz - \frac{1}{2\kappa_N^2}(c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})\ln(\lambda_r^{-1} z_h)
\end{aligned}$$

bulk	$-2r^4$	$-\frac{\phi_s^2}{6}r^2$	$\frac{(1+12gm-6\gamma^4)\phi_s^4}{12}\ln(r)$	
$\sqrt{-h}(2K)$	$8r^4$	$-\frac{2\phi_s^2}{3}r^2$	$-\frac{(1+12gm-6\gamma^4)\phi_s^4}{3}\ln(r)$	$-2(c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})B^2\ln(r)$
$\sqrt{-h}(-6)$	$-4r^4$			$\frac{3}{2}(c_0 e^{-c_1^2 c_2^2} + c_3 e^{-c_4^2 c_5^2})B^2\ln(r)$
$\sqrt{-h}F_{\mu\nu}F^{\mu\nu}$				
$\sqrt{-h}\phi(r)^2$		$\phi_s^2 r^2$	$-\frac{(1+12gm-6\gamma^4)\phi_s^4}{3}\ln(r)$	
$\sqrt{-h}\phi(r)^4$				