Note on Gauss-Bonnet-Dilaton Gravity

Contents

1 Holographic model

We begin with the five dimensional Einstein-Gauss-Bonnet theory with a dilaton theory

$$S=\frac{1}{2\kappa_N^2}\int d^5x\sqrt{-g}\left[\mathcal{R}-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-\frac{Z(\phi)}{4}F_{\mu\nu}F^{\mu\nu}-V(\phi)+\alpha H(\phi)\mathcal{R}_{GB}^2\right], \tag{1.1}$$

where $\mathcal{R}_{\rm GB}^2=\mathcal{R}^2-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta}$, and $\mathcal{R}$ is the Ricci scalar with respect to the spacetime metric $g_{\mu\nu}$, ϕ is the scalar field with a potential $V(\phi)$ and coupling functions $Z(\phi)$, $H(\phi)$ depending only on ϕ, and α is the Gauss-Bonnet coupling constant. The action yields the following field equations

$$\nabla_\mu\nabla^\mu\phi-\frac{\partial_\phi Z}{4}F_{\mu\nu}F^{\mu\nu}-\partial_\phi V+\alpha\partial_\phi H\mathcal{R}_{\rm GB}^2=0, \tag{1.2}$$

$$\nabla^\nu\left(ZF_{\nu\mu}\right)=0, \tag{1.3}$$

$$\mathcal{R}_{\mu\nu}-\frac{1}{2}\mathcal{R}g_{\mu\nu}+\Gamma_{\mu\nu}=\frac{1}{2}\partial_\mu\phi\partial_\nu\phi+\frac{Z}{2}F_{\mu\rho}F_\nu^\rho+\frac{1}{2}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-\frac{Z}{4}F_{\mu\nu}F^{\mu\nu}-V\right)g_{\mu\nu}. \tag{1.4}$$

$\Gamma_{\mu\nu}$ is defined by

$$\begin{aligned}\Gamma_{\mu\nu}=&-\mathcal{R}\left(\nabla_\mu\Psi_\nu+\nabla_\nu\Psi_\mu\right)-4\nabla^\alpha\Psi_\alpha\left(\mathcal{R}_{\mu\nu}-\frac{1}{2}\mathcal{R}g_{\mu\nu}\right)+4\mathcal{R}_{\mu\alpha}\nabla^\alpha\Psi_\nu+4\mathcal{R}_{\nu\alpha}\nabla^\alpha\Psi_\mu\\&-4g_{\mu\nu}\mathcal{R}^{\alpha\beta}\nabla_\alpha\Psi_\beta+4\mathcal{R}^\beta_{\mu\alpha\nu}\nabla^\alpha\Psi_\beta\end{aligned} \tag{1.5}$$

with

$$\Psi_\mu=\alpha\partial_\phi H\nabla_\mu\phi \tag{1.6}$$

In what follows we will specify these two functions as

$$\begin{aligned}V(\phi)&=-12\cosh\left[c_1\phi\right]+\left(6c_1^2-\frac{3}{2}\right)\phi^2+c_2\phi^6,\\Z(\phi)&=\frac{1}{1+c_3}\,\mathrm{sech}\left[c_4\phi^3\right]+\frac{c_3}{1+c_3}e^{-c_5\phi},\end{aligned} \tag{1.7}$$

As the dual system lives in a spatial plane, we choose the Poincaré coordinates with r the radial direction in the bulk. The metric ansatz reads

$$ds^2 = -f(r)e^{-\eta(r)}dt^2 + \frac{dr^2}{f(r)} + r^2\left(dx^2 + dy^2 + dz^2\right) \tag{1.8}$$

We denote the event horizon as r_h at which f vanishes. Then the temperature and the entropy density are given by

$$T = \frac{1}{4\pi} f'(r_h)\, e^{-\eta(r_h)/2}, \quad s = \frac{2\pi}{\kappa_N^2} r_h^3 \tag{1.9}$$

Substituting the ansatz into (1.2), (1.3), (1.4), we obtain the following independent equations of motion,

$$\begin{aligned}
&-\frac{12\alpha f(r)^2 H'(\phi(r))}{r^4} - \frac{6\alpha f(r)V(\phi(r))H'(\phi(r))}{r^2} + \frac{3e^{\eta(r)}\alpha f(r)Z(\phi(r))A'(r)^2H'(\phi(r))}{r^2} - \frac{48\alpha^2 f(r)^2 H(\phi(r))f'(r)H'(\phi(r))}{r^5}\\
&-\frac{48\alpha^2 f(r)^2 f'(r)H'(\phi(r))^2\phi'(r)}{r^4} - \frac{15\alpha f(r)^2H'(\phi(r))\phi'(r)^2}{r^2} - \frac{4\alpha f(r)f'(r)H'(\phi(r))\phi'(r)^2}{r} - \frac{1}{2}V'(\phi(r))\\
&+\frac{2\alpha f(r)H(\phi(r))V'(\phi(r))}{r^2} + \frac{4\alpha f(r)H'(\phi(r))\phi'(r)V'(\phi(r))}{r} + \frac{1}{4}e^{\eta(r)}A'(r)^2Z'(\phi(r))\\
&-\frac{e^{\eta(r)}\alpha f(r)H(\phi(r))A'(r)^2Z'(\phi(r))}{r^2} - \frac{2e^{\eta(r)}\alpha f(r)A'(r)^2H'(\phi(r))\phi'(r)Z'(\phi(r))}{r}\\
&+\frac{24\alpha^2 f(r)^3 H(\phi(r))H'(\phi(r))\eta'(r)}{r^5} + \frac{6\alpha f(r)^2H'(\phi(r))\eta'(r)}{r^3} - \frac{6\alpha f(r)f'(r)H'(\phi(r))\eta'(r)}{r^2} - \frac{1}{4}f(r)\phi'(r)\eta'(r)\\
&+\frac{\alpha f(r)^2H(\phi(r))\phi'(r)\eta'(r)}{r^2} - \frac{24\alpha^2 f(r)^3H'(\phi(r))^2\phi'(r)\eta'(r)}{r^4} + \frac{2\alpha f(r)^2H'(\phi(r))\phi'(r)^2\eta'(r)}{r}\\
&+\frac{48\alpha^2 f(r)^3H'(\phi(r))\phi'(r)^2H''(\phi(r))}{r^4} + \frac{48\alpha^2 f(r)^2f'(r)H'(\phi(r))\phi'(r)^2H''(\phi(r))}{r^3} - \frac{48\alpha^2 f(r)^3H'(\phi(r))\phi'(r)^2\eta'(r)H''(\phi(r))}{r^3}\\
&+\frac{1}{2}f(r)\phi''(r) - \frac{2\alpha f(r)^2H(\phi(r))\phi''(r)}{r^2} + \frac{48\alpha^2 f(r)^3H'(\phi(r))^2\phi''(r)}{r^4} + \frac{48\alpha^2 f(r)^2f'(r)H'(\phi(r))^2\phi''(r)}{r^3}\\
&-\frac{4\alpha f(r)^2H'(\phi(r))\phi'(r)\phi''(r)}{r} - \frac{48\alpha^2 f(r)^3H'(\phi(r))^2\eta'(r)\phi''(r)}{r^3}\} = 0;
\end{aligned} \tag{1.10}$$

$$\partial_r\left(e^{\eta/2}r^3ZA'(r)\right) = 0; \tag{1.11}$$

$$\begin{aligned}
&\frac{1}{2}\phi'(r)^2 - \frac{6\alpha f(r)H(\phi(r))\eta'(r)}{r^3} + \frac{3\eta'(r)}{2r} - \frac{18\alpha f(r)H'(\phi(r))\phi'(r)\eta'(r)}{r^2}\\
&-\frac{12\alpha f(r)\phi'(r)^2H''(\phi(r))}{r^2} - \frac{12\alpha f(r)H'(\phi(r))\phi''(r)}{r^2} = 0;
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
&\frac{6f(r)}{r^2} + V(\phi(r)) + \frac{1}{2}e^{\eta(r)}Z(\phi(r))A'(r)^2 - \frac{12\alpha f(r)H(\phi(r))f'(r)}{r^3} + \frac{3f'(r)}{r} - \frac{24\alpha f(r)^2H'(\phi(r))\phi'(r)}{r^3}\\
&-\frac{36\alpha f(r)f'(r)H'(\phi(r))\phi'(r)}{r^2} + \frac{6\alpha f(r)^2H(\phi(r))\eta'(r)}{r^3} - \frac{3f(r)\eta'(r)}{2r} + \frac{18\alpha f(r)^2H'(\phi(r))\phi'(r)\eta'(r)}{r^2}\\
&-\frac{12\alpha f(r)^2\phi'(r)^2H''(\phi(r))}{r^2} - \frac{12\alpha f(r)^2H'(\phi(r))\phi''(r)}{r^2} = 0;
\end{aligned} \tag{1.13}$$

It is worth noting that here we need a simple distinction: $G'(\phi(r)) = \partial_\phi G$, $G'(r) = \partial_r G$

2 UV expansion

We introduce a new coordinate $z = 1/r$ and functions: $\phi(r) = \Phi(z)/r$, $f(r) = r^2F(z)$, $\eta(r) = \Sigma(z)$, $A(r) = A(z)$. Near the AdS boundary $r \to \infty$(or $z \to 0$) where $\phi \to 0$, we obtain the following asymptotic expansion:

$$
\begin{aligned}
\Phi(z) &=: \phi_s + \phi_{uv_1} z + \phi_{log_1} z \log z + \phi_v z^2 + \phi_{log_2} z^2 \log z + \sum_{i=3}^{5} (\phi_{uv_i} z^i + \phi_{log_i} z^i \log z); \\
F(z) &=: f_c + \sum_{i=1}^{5} (f_{uv_i} z^i + f_{log_i} z^i \log z); \\
\Sigma(z) &=: \eta_0 + \sum_{i=1}^{5} (\eta_{uv_i} z^i + \eta_{log_i} z^i \log z); \\
A(z) &=: \mu + a_{uv_1} z + a_{log_1} z \log z - \frac{1}{2}\rho_n z^2 + a_{log_2} z^2 \log z + \sum_{i=3}^{5} (a_{uv_i} z^i + a_{log_i} z^i \log z);
\end{aligned}
\tag{2.1}
$$

Substituting the expansions into the equations of motion give the expansion coefficients (See Program for details). In addition, $H(\phi)$ satisfies the following conditions

$$
H(0) = 0, H'(0) = 0, H''(0) = \frac{H(0)}{20} = 0, \tag{2.2}
$$

$$
H^{(4)}(0) = \frac{1 - 6c_1^4 - 21600\alpha^2 H^{(3)}(0)^2}{60\alpha}. \tag{2.3}
$$

3 IR expansion

The smoothness of the event horizon yields the following analytic expansion in terms of $(r - r_h)$ in the IR:

$$
\begin{aligned}
f &= f_h\,(r - r_h) + \ldots \\
\eta &= \eta_h^0 + \eta_h^1\,(r - r_h) + \ldots \\
A &= a_h\,(r - r_h) + \ldots \\
\phi &= \phi_h^0 + \phi_h^1\,(r - r_h) + \ldots
\end{aligned}
\tag{3.1}
$$

4 Thermodynamics

We now compute the free energy density Ω, which is identified as the temperature T times the renormalized action in the Euclidean signature. Since we consider a stationary problem, the Euclidean action is related to the Minkowski one by a minus sign. Moreover, we should include the Gibbons-Hawking boundary term for a well-defined Dirichlet variational principle and a surface counterterm for removing divergence. Therefore, we have:

$$
-\Omega V = T(S - S_\partial)_{on-shell}, \tag{4.1}
$$

with $V = \int dxdydz$ and $t \in [0, \beta]$.

The boundary terms take the form

$$S_\partial = \frac{1}{2\kappa_N^2}\int_{r\to\infty} dx^4\sqrt{-h}[(2K-6+4\alpha H(\phi)J)$$
$$-\frac{1}{2}\phi^2 - \frac{6c_1^4-1}{12}\phi^4\ln(r) - b\phi^4 + \frac{1}{4}F_{\rho\lambda}F^{\rho\lambda}\ln(r)$$
$$-\frac{44}{3}\alpha H^{(3)}(\phi)\phi^3 - \left(1800\alpha^2 H^{(3)}(\phi)^2 + 5\alpha H^{(4)}(\phi)\right)\phi^4\ln(r)], \tag{4.2}$$

where J is the trace of

$$J_{ab} = \frac{1}{3}\left(2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^cK_{db} - K^2K_{ab}\right), \tag{4.3}$$

h_{ab} is the induced metric at the AdS boundary and K_{ab} is the extrinsic curvature defined by the outward pointing normal vector to the boundary.

The energy-momentum tensor of the dual boundary theory reads

$$\begin{aligned} T_{ab} &= \lim_{r\to\infty}\frac{2r^2}{\sqrt{-\det h}}\frac{\delta\left(S+S_\partial\right)_{\text{on-shell}}}{\delta h^{ab}} \\ &= \frac{1}{2\kappa_N^2}\lim_{r\to\infty} r^2\left[2\left(Kh_{ab}-K_{ab}-3h_{ab}\right)\right. \\ &\quad + 4\alpha H(\phi)\left(3J_{ab}-Jh_{ab}\right) \\ &\quad + 4\alpha\left[2K_{ea}K_b^e - 2KK_{ab} + h_{ab}(K^2-K_{cd}K^{cd})\right]n^e\partial_e H(\phi) \\ &\quad - \left(\frac{1}{2}\phi^2 + \frac{6c_1^4-1}{12}\phi^4\ln(r) + b\phi^4 + \frac{44}{3}\alpha H^{(3)}(\phi)\phi^3\right. \\ &\quad \left. + \left(1800\alpha^2H^{(3)}(\phi)^2 + 5\alpha H^{(4)}(\phi)\right)\phi^4\ln(r)\right)h_{ab} \\ &\quad \left. - \left(F_{ac}F_b^c - \frac{1}{4}h_{ab}F_{cd}F^{cd}\right)\ln(r)\right]. \end{aligned} \tag{4.4}$$

we obtain

$$\epsilon := T_{tt} = \frac{\phi_s^4}{96\kappa_N^2} + \frac{b\phi_s^4}{2\kappa_N^2} + \frac{\phi_s\phi_v}{2\kappa_N^2} - \frac{3f_{uv_4}}{2\kappa_N^2} + \frac{900\alpha^2\phi_s^4H^{(3)}(0)^2}{\kappa_N^2} - \frac{11\alpha\phi_s^4H^{(4)}(0)}{6\kappa_N^2} \tag{4.5}$$

$$P := \frac{\phi_s^4}{32\kappa_N^2} - \frac{b\phi_s^4}{2\kappa_N^2} - \frac{c_1^4\phi_s^4}{12\kappa_N^2} + \frac{\phi_s\phi_v}{2\kappa_N^2} - \frac{f_{uv_4}}{2\kappa_N^2} - \frac{1200\alpha^2\phi_s^4H^{(3)}(0)^2}{\kappa_N^2} - \frac{7\alpha\phi_s^4H^{(4)}(0)}{3\kappa_N^2} \tag{4.6}$$

An other useful radially conserved quantity reads

$$\mathcal{Q}_{GB} = \frac{1}{2\kappa_N^2}r^3e^{\eta/2}\left[\left(r^2+4\alpha fH(\phi)+8\alpha rfH'(\phi)\phi'\right)\left(\frac{f}{r^2}e^{-\eta}\right)' - ZA_tA_t'\right] \tag{4.7}$$

Employing the equations of motion, we obtain

$$\begin{aligned} \Omega =& \frac{1}{2\kappa_N^2}\lim_{r\to\infty}[2e^{-\eta/2}r^2f + 8\alpha e^{-\eta/2}f(-rf'+f(r\eta'-1))(H(\phi)-rH'(\phi)\phi')] \\ &- e^{-\eta/2}r^3\sqrt{f}[(2K-6+4\alpha H(\phi)J) - \frac{1}{2}\phi^2 - \frac{6c_1^4-1}{12}\phi^4\ln(r) - b\phi^4 \\ &- \frac{44}{3}\alpha H^{(3)}(\phi)\phi^3 - \left(1800\alpha^2H^{(3)}(\phi)^2 + 5\alpha H^{(4)}(\phi)\right)\phi^4\ln(r)], \end{aligned} \tag{4.8}$$

5 Shear Viscosity to entropy density ratio

Kubo's formula relates the shear viscosity to the low frequency and zero momentum limit of the retarded Green's function of the stress tensor in the CFT

$$G^R_{xy,xy}(\omega, \mathbf{k}=0) = -i \int dtd\mathbf{x} e^{i\omega t}\theta(t) \langle [T_{xy}(x), T_{xy}(0)] \rangle \tag{5.1}$$

Concretely one has

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \operatorname{Im} G^R_{xy,xy}(\omega, \mathbf{k}=0). \tag{5.2}$$

Translating the calculation of the correlator to a holographic one, one first finds the effective action for the metric perturbation $h^y_x(t,r) = \int \frac{d^4k}{(2\pi)^4} \phi_k(r) e^{-i\omega t + ikz}$. Evaluating the action (1.1) to quadratic order in the fluctuations $\phi_k(r)$ yields

$$\begin{aligned} I^{(2)}_\phi &= \tfrac{1}{2\kappa_N^2} \int \tfrac{d^4k}{(2\pi)^4} dr \left(A(r)\phi''_k \phi_{-k} + B(r)\phi'_k \phi'_{-k} + C(r)\phi'_k \phi_{-k} \right. \\ &\left. + D(r)\phi_k \phi_{-k} + E(r)\phi''_k \phi''_{-k} + F(r)\phi''_k \phi'_{-k} \right) + \mathcal{K}, \end{aligned} \tag{5.3}$$

we have also added a generalized Gibbons-Hawking boundary term $\mathcal{K}$ in (5.3),

$$\mathcal{K} = \frac{1}{2\kappa_N^2} \int \frac{d^4k}{(2\pi)^4} \left(K_1 + K_2 + K_3\right)\Big|_{r=r_h}^{r=\infty}. \tag{5.4}$$

with

$$\begin{aligned} K_1 &= -A\phi'_k \phi_{-k} \qquad\qquad K_2 = -\tfrac{F}{2}\phi'_k \phi'_{-k} \\ K_3 &= E\left(p_1 \phi'_k + 2p_0 \phi_k\right)\phi'_{-k} \end{aligned} \tag{5.5}$$

Hence we arrive at the result

$$\eta = \lim_{\omega \to 0} \frac{\Pi(r)}{i\omega \phi(r)} = \frac{1}{\kappa_N^2}\left(\kappa_2(r_h) + \kappa_4(r_h)\right). \tag{5.6}$$

where in the second expression, we have evaluated the ratio at the horizon $r = r_h$ and defined the quantities

$$\kappa_2(r) = \sqrt{-\frac{g_{rr}(r)}{g_{tt}(r)}} \left(A(r) - B(r) + \frac{F'(r)}{2} \right), \quad \kappa_4(r) = \left(E(r) \left(\sqrt{-\frac{g_{rr}(r)}{g_{tt}(r)}} \right)' \right)'. \tag{5.7}$$

It is straightforward to obtain the shear viscosity:

$$\eta_{GB} = \frac{r_h^2}{2\kappa_N^2} \left[r_h - 2\alpha f'(r_h)(H(\phi(r_h)) + r_h H'(\phi(r_h))\phi'(r_h)) \right]. \tag{5.8}$$

So we have

$$\frac{\eta_{GB}}{s} = \frac{1}{4\pi} \left[1 - \frac{2\alpha f'(r_h)}{r_h} (H(\phi(r_h)) + r_h H'(\phi(r_h))\phi'(r_h)) \right]. \tag{5.9}$$