General Topology

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1. metric spaces, open balls, open sets, notion of a topology, differentmetrics may define the same topology, some topologies cannot come from metrics
2. closed sets, epsilon-delta definition of continuity in metric spaces, continuity of functions between topological spaces, subspace topology, bases
3. Composites of continuous maps are continuous, subbases, topology generated by a subset of the power set, Cartesian products of sets
4. Products of pairs of topological spaces, homeomorphism, not every continuous bijection is a homeomorphism, uniqueness of product topology
5. Products of families of topological spaces and their universal properties, box topology, open and closed maps, coproducts
6. Quotient topology, limit points
7. Closure of a subset, limit points of a subset, convergence of sequences, limits of sequences lie in the closure but not conversely, limits need not be unique
8. Interior of a subset, boundary. Interaction of closure, interior, boundary and complements. First countable topological spaces
9. Preorders, directed sets, nets, convergence of nets, points in the closure and limits of nets, continuity and convergence of nets, being Hausdorff and uniqueness of limits of nets
10. Subnets, subnet of a convergent net. Compactness: images of compact sets are compact, closed subsets of compact sets are compact, compact subsets of Hausdorff spaces are closed, if f: $X \to Y$ is a continuous bijection, X compact, Y Hausdorff, then f is a homeomorphism
11. Bolzano-Weirstrass, tube lemma, products of two compact spaces are compact, a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded; X compact, $X \to \mathbb{R}$ continuous $\Rightarrow$ f achieves max and min on X
12. Finite Intersection Property (FIP) and compactness in terms of FIP, cluster/limit/accumilation point of a net, a net has a cluster point iff it has a convergent subnet, a space is compact iff a net has a cluster point iff a net has a convergent subnet
13. Tychonoff's theorem: a product of compact spaces is compact. Lebesque lemma

14. A metric space is compact iff every sequence has a convergent subsequence iff the space is complete and totally bounded. Separation axioms

15. A Hausdorff space X is regular iff for every x in X any nbd N of x contains a closed nbd of x. There is a Hausdorff space which is not regular. There are also regular spaces that are not normal. Metric spaces are normal

16. Compact Hausdorff spaces are normal, completely regular and Tychonoff spaces. Urysohn's lemma

17. $[0,1]^N$ is metrizable. Urysohn's metrization theorem: second countable completely regular T_1 spaces embed in $[0,1]^N$ hence are metrizable

18. 2nd countable + regular $\Rightarrow$ metrizable, Lindelof spaces, 2nd countable $\Rightarrow$ Lindelof. Tietze extension theorem

19. Moore plane is not normal. Local compactness. Manifolds. LCH (locally compact Hausdorff). 2nd countable LCH space is normal and metrizable. Compactifications and 1 point compactifications

20. X has a 1 point compactification $\Leftrightarrow$ X is LCH and noncompact. 1 point compactifications are unique. Proper maps. Continuous maps need not extend to continuous maps on 1 point compactifications

21. a map f extends to a continuos map of 1-point compactifications $\Leftrightarrow$ f is proper. Proper continuous maps between LCH spaces are closed. Topological groups, continuous group actions, orbit spaces, proper group actoins

22. Quotients of LCH groups acting on LCH spaces are Hausdorff. Notion of connectedness. [0,1] is connected. X is connected $\Leftrightarrow$ any continuous map from X to any discrete space is constant

23. Connected components of a space are connected and closed. A in X connected and E sits between A and the closure of A then E is connected. Path connected $\Rightarrow$ connected but there are connected spaces that are not path connected

24. Path components. Connected and locally path connected spaces are connected. Manifolds are locally path connected. Notions of partition of unity and paracompactness. A compact Hausdorff manifold may be embedded in some $\mathbb{R}^N$

25. sigma-compactness. Locally compact sigma-compact Hausdorff spaces is paracompact. Paracompact spaces are normal

26. Existence of partitions of 1 on a paracompact space. A manifold M is paracompact iff M is a disjoint union of Hausdorff second countable manifolds

27. Homotopy; homotopy classes of maps compose. Notion of a categroy
28. Isomorphisms in a category. Homotopy equivalence of spaces. Functors. Groupoids
29. Construction of the fundamental groupoid of a space. Fundamental groups. Fundamental groupoid of a convex subset of $\mathbb{R}^n$. Pair groupoid
30. The functor Π from spaces to groupoids. Natural transformations. Homotopies give rise to natural transformations
31. Natural isomorphisms. Equivalent categories. Full, faithful and essentially surjective functors. Equivalences of categories are full, faithful and essentially surjective
32. A fully faithful and essentially surjective functor is part of the equivalences of categories. Pushouts
33. Uniqueness of pushouts. A space is a pushout of its cover. Statement of Brown-Seifert-van Kampen: fundamental groupoid functor takes pushouts in Top to pushouts in Groupoid
34. Computation of the fundamental groupoid of the circle and the fundamental group of the circle
35. Proof of B-S-v K theorem. Free products of groups
36. Pushouts in the category of groups = amalgamated free products. Proof of Seifert - van Kampen from B.-S. v. K
37. Degree of a map from the circle to the circle. Fundamental Theorem of Algebra. Definition of a compact-open topology
38. compact-open topology
39. uniform convergence on compact sets and compact-open topology
40. Stone-Čech compactification

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