Solar Force-Free Magnetic Fields

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$(x, y, z) \overset{\textit{slide down along z axis}}{\rightarrow} (x', y', z') \overset{\textit{clockwise rotation by an angle of } \Phi}{\rightarrow} (X, Y, Z).$

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} = \begin{bmatrix} x & y & z+l \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} X & Y & Z \end{bmatrix} = \begin{bmatrix} x & y & z+l \end{bmatrix} \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} X & Y & Z \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\phi & r\sin\theta\sin\phi & r\cos\theta \end{bmatrix} \tag{3}$$

where $r \geqslant 0, 0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi < 2\pi$.

From (3)

$$X^2 + Y^2 + Z^2 = r^2, X^2 + Y^2 = r^2\sin^2\theta, \left(X^2 + Y^2\right)^{\frac{1}{2}} = r\sin\theta, (\because 0 \leqslant \theta \leqslant \pi, \therefore \sin\theta \geqslant 0) \tag{4}$$

Therefor,

$$\frac{Z}{(X^2 + Y^2 + Z^2)^{\frac{1}{2}}} = \cos\theta, \tag{5}$$

and

$$\begin{aligned} \frac{X}{(X^2 + Y^2)^{\frac{1}{2}}} &= \cos\phi \\ \frac{Y}{(X^2 + Y^2)^{\frac{1}{2}}} &= \sin\phi \end{aligned} \tag{6}$$

1. $0 \leqslant \phi < \pi$,

$$\frac{Y}{(X^2 + Y^2)^{\frac{1}{2}}} = \sin\phi \geqslant 0 \Rightarrow \phi = \cos^{-1}\left(\frac{X}{(X^2 + Y^2)^{\frac{1}{2}}}\right) \in [0, \pi) \tag{7}$$

2. $\pi \leqslant \phi < 2\pi$,

$$\frac{Y}{(X^2 + Y^2)^{\frac{1}{2}}} = \sin\phi \leqslant 0 \Rightarrow \phi = 2\pi - \cos^{-1}\left(\frac{X}{(X^2 + Y^2)^{\frac{1}{2}}}\right) \in [\pi, 2\pi] \tag{8}$$

If (x, y, z) is given, then we can get (r, θ, ϕ).

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If $\vec{r} = u_1\vec{e_1} + u_2\vec{e_2} + u_3\vec{e_3} = u_4\vec{e_4} + u_5\vec{e_5} + u_6\vec{e_6}$, and $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}\{\vec{e_4}, \vec{e_5}, \vec{e_6}\}$ is an orthogonal curvilinear coordinate system, respectively. Then

$$\vec{e_4} = \frac{1}{h_4}\left(\frac{\partial u_1}{\partial u_4}\vec{e_1} + \frac{\partial u_2}{\partial u_4}\vec{e_2} + \frac{\partial u_3}{\partial u_4}\vec{e_3}\right) \tag{9}$$

$$\vec{e_5} = \frac{1}{h_5}\left(\frac{\partial u_1}{\partial u_5}\vec{e_1} + \frac{\partial u_2}{\partial u_5}\vec{e_2} + \frac{\partial u_3}{\partial u_5}\vec{e_3}\right) \tag{10}$$

$$\vec{e_6} = \frac{1}{h_6}\left(\frac{\partial u_1}{\partial u_6}\vec{e_1} + \frac{\partial u_2}{\partial u_6}\vec{e_2} + \frac{\partial u_3}{\partial u_6}\vec{e_3}\right) \tag{11}$$

Form (9),(10),(11), we can get $h_i \quad (4 \le i \le 6)$,

$$h_i = \sqrt{\left(\frac{\partial u_1}{\partial u_i}\right)^2 + \left(\frac{\partial u_2}{\partial u_i}\right)^2 + \left(\frac{\partial u_3}{\partial u_i}\right)^2} \tag{12}$$

(9),(10),(11) also can be written as a matrix form as flowing:

$$\begin{bmatrix}\vec{e_4}\\ \vec{e_5}\\ \vec{e_6}\end{bmatrix} = \begin{bmatrix}\frac{1}{h_4}\frac{\partial u_1}{\partial u_4} & \frac{1}{h_4}\frac{\partial u_2}{\partial u_4} & \frac{1}{h_4}\frac{\partial u_3}{\partial u_4}\\ \frac{1}{h_5}\frac{\partial u_1}{\partial u_5} & \frac{1}{h_5}\frac{\partial u_2}{\partial u_5} & \frac{1}{h_5}\frac{\partial u_3}{\partial u_5}\\ \frac{1}{h_6}\frac{\partial u_1}{\partial u_6} & \frac{1}{h_6}\frac{\partial u_2}{\partial u_6} & \frac{1}{h_6}\frac{\partial u_3}{\partial u_6}\end{bmatrix}\begin{bmatrix}\vec{e_1}\\ \vec{e_2}\\ \vec{e_3}\end{bmatrix} \tag{13}$$

where $h_i = \left|\frac{\partial \vec{r}}{\partial u_i}\right|, i = 4, 5, 6.$

Then

$$\begin{aligned}\vec{B} &= \begin{bmatrix}B_1 & B_2 & B_3\end{bmatrix}\begin{bmatrix}\vec{e_1}\\ \vec{e_2}\\ \vec{e_3}\end{bmatrix}\\ &= \begin{bmatrix}B_4 & B_5 & B_6\end{bmatrix}\begin{bmatrix}\vec{e_4}\\ \vec{e_5}\\ \vec{e_6}\end{bmatrix}\\ &= \begin{bmatrix}B_4 & B_5 & B_6\end{bmatrix}\begin{bmatrix}\frac{1}{h_4}\frac{\partial u_1}{\partial u_4} & \frac{1}{h_4}\frac{\partial u_2}{\partial u_4} & \frac{1}{h_4}\frac{\partial u_3}{\partial u_4}\\ \frac{1}{h_5}\frac{\partial u_1}{\partial u_5} & \frac{1}{h_5}\frac{\partial u_2}{\partial u_5} & \frac{1}{h_5}\frac{\partial u_3}{\partial u_5}\\ \frac{1}{h_6}\frac{\partial u_1}{\partial u_6} & \frac{1}{h_6}\frac{\partial u_2}{\partial u_6} & \frac{1}{h_6}\frac{\partial u_3}{\partial u_6}\end{bmatrix}\begin{bmatrix}\vec{e_1}\\ \vec{e_2}\\ \vec{e_3}\end{bmatrix}\end{aligned} \tag{14}$$

Finally,

$$\begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix} = \begin{bmatrix} B_4 & B_5 & B_6 \end{bmatrix} \begin{bmatrix} \frac{1}{h_4}\frac{\partial u_1}{\partial u_4} & \frac{1}{h_4}\frac{\partial u_2}{\partial u_4} & \frac{1}{h_4}\frac{\partial u_3}{\partial u_4} \\ \frac{1}{h_5}\frac{\partial u_1}{\partial u_5} & \frac{1}{h_5}\frac{\partial u_2}{\partial u_5} & \frac{1}{h_5}\frac{\partial u_3}{\partial u_5} \\ \frac{1}{h_6}\frac{\partial u_1}{\partial u_6} & \frac{1}{h_6}\frac{\partial u_2}{\partial u_6} & \frac{1}{h_6}\frac{\partial u_3}{\partial u_6} \end{bmatrix} \tag{15}$$

1. When

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}, \begin{bmatrix} B_4 \\ B_5 \\ B_6 \end{bmatrix} = \begin{bmatrix} B_{x'} \\ B_{y'} \\ B_{z'} \end{bmatrix} \tag{16}$$

$$\vec{e_4} = \frac{1}{h_4}\left(\frac{\partial u_1}{\partial u_4}\vec{e_1} + \frac{\partial u_2}{\partial u_4}\vec{e_2} + \frac{\partial u_3}{\partial u_4}\vec{e_3}\right) = \frac{1}{h_4}\left(\frac{\partial x}{\partial x'}\vec{e_1} + \frac{\partial y}{\partial x'}\vec{e_2} + \frac{\partial z}{\partial x'}\vec{e_3}\right) = \frac{1}{h_4}\vec{e_1} \Rightarrow h_4 = 1 \tag{17}$$

$$\vec{e_5} = \frac{1}{h_5}\left(\frac{\partial u_1}{\partial u_5}\vec{e_1} + \frac{\partial u_2}{\partial u_5}\vec{e_2} + \frac{\partial u_3}{\partial u_5}\vec{e_3}\right) = \frac{1}{h_5}\left(\frac{\partial x}{\partial y'}\vec{e_1} + \frac{\partial y}{\partial y'}\vec{e_2} + \frac{\partial z}{\partial y'}\vec{e_3}\right) = \frac{1}{h_5}\vec{e_2} \Rightarrow h_5 = 1 \tag{18}$$

$$\vec{e_6} = \frac{1}{h_6}\left(\frac{\partial u_1}{\partial u_6}\vec{e_1} + \frac{\partial u_2}{\partial u_6}\vec{e_2} + \frac{\partial u_3}{\partial u_6}\vec{e_3}\right) = \frac{1}{h_5}\left(\frac{\partial x}{\partial z'}\vec{e_1} + \frac{\partial y}{\partial z'}\vec{e_2} + \frac{\partial z}{\partial z'}\vec{e_3}\right) = \frac{1}{h_6}\vec{e_3} \Rightarrow h_6 = 1 \tag{19}$$

$$\begin{bmatrix} B_x & B_y & B_z \end{bmatrix} = \begin{bmatrix} B_{x'} & B_{y'} & B_{z'} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

2. When

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} B_{x'} \\ B_{y'} \\ B_{z'} \end{bmatrix}, \begin{bmatrix} B_4 \\ B_5 \\ B_6 \end{bmatrix} = \begin{bmatrix} B_X \\ B_Y \\ B_Z \end{bmatrix} \tag{21}$$

by (2)

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} X\cos\Phi + Z\sin\Phi \\ Y \\ -X\sin\Phi + Z\cos\Phi \end{bmatrix} \tag{22}$$

$$\begin{aligned} \vec{e_4} &= \frac{1}{h_4}\left[\frac{\partial u_1}{\partial u_4}\vec{e_1} + \frac{\partial u_2}{\partial u_4}\vec{e_2} + \frac{\partial u_2}{\partial u_4}\vec{e_3}\right] \\ &= \frac{1}{h_4}\left[\frac{\partial (X\cos\Phi + Z\sin\Phi)}{\partial X}\vec{e_1} + \frac{\partial Y}{\partial X}\vec{e_2} + \frac{\partial (-X\sin\Phi + Z\cos\Phi)}{\partial X}\vec{e_3}\right] \\ &= \frac{1}{h_4}\left[\cos\Phi\vec{e_1} - \sin\Phi\vec{e_3}\right] \Rightarrow h_4 = 1 \end{aligned} \tag{23}$$

$$\begin{aligned} \vec{e_5} &= \frac{1}{h_5}\left[\frac{\partial u_1}{\partial u_5}\vec{e_1} + \frac{\partial u_2}{\partial u_5}\vec{e_2} + \frac{\partial u_2}{\partial u_5}\vec{e_3}\right] \\ &= \frac{1}{h_5}\left[\frac{\partial (X\cos\Phi + Z\sin\Phi)}{\partial Y}\vec{e_1} + \frac{\partial Y}{\partial Y}\vec{e_2} + \frac{\partial (-X\sin\Phi + Z\cos\Phi)}{\partial Y}\vec{e_3}\right] \\ &= \frac{1}{h_5}\left[\vec{e_2}\right] \Rightarrow h_5 = 1 \end{aligned} \tag{24}$$

$$
\begin{aligned}
\vec{e_6} &= \frac{1}{h_6}\left[\frac{\partial u_1}{\partial u_6}\vec{e_1} + \frac{\partial u_2}{\partial u_6}\vec{e_2} + \frac{\partial u_2}{\partial u_6}\vec{e_3}\right] \\
&= \frac{1}{h_6}\left[\frac{\partial\left(X\cos\Phi + Z\sin\Phi\right)}{\partial Z}\vec{e_1} + \frac{\partial Y}{\partial Z}\vec{e_2} + \frac{\partial\left(-X\sin\Phi + Z\cos\Phi\right)}{\partial Z}\vec{e_3}\right] \\
&= \frac{1}{h_6}\left[\sin\Phi\vec{e_1} + \cos\Phi\vec{e_3}\right] \Rightarrow h_6 = 1
\end{aligned} \tag{25}
$$

$$
\begin{bmatrix} B_{x'} & B_{y'} & B_{z'} \end{bmatrix} = \begin{bmatrix} B_X & B_Y & B_Z \end{bmatrix} \begin{bmatrix} \cos\Phi & 0 & -\sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & \cos\Phi \end{bmatrix} \tag{26}
$$

3. When

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} B_X \\ B_Y \\ B_Z \end{bmatrix}, \begin{bmatrix} B_4 \\ B_5 \\ B_6 \end{bmatrix} = \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix} \tag{27}
$$

$$
\begin{aligned}
\vec{e_4} &= \frac{1}{h_4}\left[\frac{\partial u_1}{\partial u_4}\vec{e_1} + \frac{\partial u_2}{\partial u_4}\vec{e_2} + \frac{\partial u_2}{\partial u_4}\vec{e_3}\right] \\
&= \frac{1}{h_4}\left[\frac{\partial\left(r\sin\theta\cos\phi\right)}{\partial r}\vec{e_1} + \frac{\partial\left(r\sin\theta\sin\phi\right)}{\partial r}\vec{e_2} + \frac{\partial\left(r\cos\theta\right)}{\partial r}\vec{e_3}\right] \\
&= \frac{1}{h_4}\left[sin\theta\cos\phi\vec{e_1} + \sin\theta\sin\phi\vec{e_2} + \cos\theta\vec{e_3}\right] \\
&\Rightarrow h_4 = 1
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\vec{e_5} &= \frac{1}{h_5}\left[\frac{\partial u_1}{\partial u_5}\vec{e_1} + \frac{\partial u_2}{\partial u_5}\vec{e_2} + \frac{\partial u_2}{\partial u_5}\vec{e_3}\right] \\
&= \frac{1}{h_5}\left[\frac{\partial\left(r\sin\theta\cos\phi\right)}{\partial\theta}\vec{e_1} + \frac{\partial\left(r\sin\theta\,\mathrm{sins}\,\phi\right)}{\partial\theta}\vec{e_2} + \frac{\partial\left(r\cos\theta\right)}{\partial\theta}\vec{e_3}\right] \\
&= \frac{1}{h_5}\left[r\cos\theta\cos\phi\vec{e_1} + r\cos\theta\sin\phi\vec{e_2} - r\sin\theta\vec{e_3}\right] \\
&\Rightarrow h_5 = r
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\vec{e_6} &= \frac{1}{h_6}\left[\frac{\partial u_1}{\partial u_6}\vec{e_1} + \frac{\partial u_2}{\partial u_6}\vec{e_2} + \frac{\partial u_2}{\partial u_6}\vec{e_3}\right] \\
&= \frac{1}{h_6}\left[\frac{\partial\left(r\sin\theta\cos\phi\right)}{\partial\phi}\vec{e_1} + \frac{\partial\left(r\sin\theta\sin\phi\right)}{\partial\phi}\vec{e_2} + \frac{\partial\left(r\cos\theta\right)}{\partial\phi}\vec{e_3}\right] \\
&= \frac{1}{h_6}\left[-r\sin\theta\sin\phi\vec{e_1} + r\sin\theta\cos\phi\vec{e_2}\right] \\
&\Rightarrow h_6 = r\sin\theta
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&\begin{bmatrix} B_X & B_Y & B_Z \end{bmatrix} \\
&= \begin{bmatrix} B_r & B_\theta & B_\phi \end{bmatrix} \begin{bmatrix} sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{1}{r}r\cos\theta\cos\phi & \frac{1}{r}r\cos\theta\sin\phi & \frac{1}{r}\left(-r\sin\theta\right) \\ \frac{1}{r\sin\theta}\left(-r\sin\theta\sin\phi\right) & \frac{1}{r\sin\theta}\left(r\sin\theta\cos\phi\right) & 0 \end{bmatrix} \\
&= \begin{bmatrix} B_r & B_\theta & B_\phi \end{bmatrix} \begin{bmatrix} sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}
\end{aligned} \tag{31}
$$

Finally, by (20),(26),(31), we can get

$$
\begin{aligned}
&\begin{bmatrix} B_x & B_y & B_z \end{bmatrix} \\
&= \begin{bmatrix} B_{x'} & B_{y'} & B_{z'} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} B_X & B_Y & B_Z \end{bmatrix} \begin{bmatrix} \cos\Phi & 0 & -\sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & \cos\Phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} B_r & B_\theta & B_\phi \end{bmatrix} \begin{bmatrix} sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos\Phi & 0 & -\sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & \cos\Phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} B_r & B_\theta & B_\phi \end{bmatrix} \begin{bmatrix} \cos\Phi sin\theta\cos\phi + \sin\Phi\cos\theta & \sin\theta\sin\phi & -\sin\Phi sin\theta\cos\phi - \sin\Phi\cos\theta \\ \cos\Phi\cos\theta\cos\phi - \sin\Phi\sin\theta & \cos\theta\sin\phi & -\sin\Phi\cos\theta\cos\phi - \cos\Phi\sin\theta \\ -\cos\Phi\sin\phi & \cos\phi & \sin\Phi\sin\phi \end{bmatrix}
\end{aligned} \tag{32}
$$

II title

$$
\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}
$$

$$
\begin{aligned}
\nabla \times \vec{B} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_y & B_z \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ B_x & B_z \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ B_x & B_y \end{vmatrix} \vec{k} \\
&= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \vec{i} - \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \right) \vec{j} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \vec{k} \\
&= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \vec{i} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \vec{j} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \vec{k}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\nabla \times \vec{B} \right) \times \vec{B} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} & \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \\ B_x & B_y & B_z \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \\ B_y & B_z \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \\ B_x & B_z \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} & \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ B_x & B_y \end{vmatrix} \vec{k} \\
&= \left[B_z \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) - B_y \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \right] \vec{i} - \left[B_z \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) - B_x \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \right] \vec{j} + \left[B_y \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) - B_x \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \right] \vec{k} \\
&= \left[B_z \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) - B_y \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \right] \vec{i} + \left[B_x \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) - B_z \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \right] \vec{j} + \left[B_y \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) - B_x \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \right] \vec{k}
\end{aligned}
$$

In PINN:

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \tag{DE1$'$}$$

$$B_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - B_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = 0 \tag{DE2$'$}$$

$$B_x\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - B_z\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = 0 \tag{DE3$'$}$$

$$B_y\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - B_x\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) = 0 \tag{DE4$'$}$$

$$\left(B_x, B_y, B_{z=0}\right) \;\; is \;\; given. \tag{BC$'$}$$

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