Solar Force-Free Magnetic Fields

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1. $\nabla \cdot \vec{B} = 0$

$$\exists \vec{A} = \left(A_r, A_\theta, A_\phi\right) s.t. \vec{B} = \nabla \times \vec{A},$$

$$\begin{aligned}
\nabla \times \vec{A} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e_1} & h_2 \vec{e_2} & h_3 \vec{e_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \\
&= \frac{1}{1 \times r \times r \sin\theta} \begin{vmatrix} 1\vec{e_r} & r\vec{e_\theta} & r\sin\theta \vec{e_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} = 0 \\ 1A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \frac{\partial}{\partial\theta} & 0 \\ rA_\theta & r\sin\theta A_\phi \end{vmatrix} \vec{e_r} - \frac{1}{r^2\sin\theta} \begin{vmatrix} \frac{\partial}{\partial r} & 0 \\ A_r & r\sin\theta A_\phi \end{vmatrix} r\vec{e_\theta} + \frac{1}{r^2\sin\theta} \begin{vmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} \\ A_r & rA_\theta \end{vmatrix} r\sin\theta \vec{e_\phi} \\
&= \frac{1}{r^2\sin\theta} \frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi\right)\vec{e_r} - \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r\sin\theta A_\phi\right)\vec{e_\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\vec{e_\phi} \\
&= \frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi\right), -\frac{\partial}{\partial r}\left(r\sin\theta A_\phi\right), \sin\theta\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\right)
\end{aligned} \tag{1}$$

2. $1 \Rightarrow \vec{B} = \frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\underbrace{\left(r\sin\theta A_\phi\right)}_{\widetilde{A}}, -\frac{\partial}{\partial r}\left(r\sin\theta A_\phi\right), \underbrace{\sin\theta\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)}_{b_\phi}\right)$

$$\begin{aligned}
\nabla \times \vec{B} &= \frac{1}{r^2\sin\theta} \begin{vmatrix} \vec{e_r} & r\vec{e_\theta} & r\sin\theta\vec{e_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} = 0 \\ \frac{1}{r\sin\theta}\frac{1}{r}\frac{\partial\widetilde{A}}{\partial\theta} & r\frac{1}{r\sin\theta}\left(-\frac{\partial\widetilde{A}}{\partial r}\right) & r\sin\theta\frac{1}{r\sin\theta}b_\phi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta} \begin{vmatrix} \vec{e_r} & r\vec{e_\theta} & r\sin\theta\vec{e_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} = 0 \\ \frac{1}{r\sin\theta}\frac{1}{r}\frac{\partial\widetilde{A}}{\partial\theta} & r\frac{1}{r\sin\theta}\left(-\frac{\partial\widetilde{A}}{\partial r}\right) & r\sin\theta\frac{1}{r\sin\theta}b_\phi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\sin\theta\frac{1}{r\sin\theta}b_\phi\right)\vec{e_r} - r\frac{\partial}{\partial r}\left(r\sin\theta\frac{1}{r\sin\theta}b_\phi\right)\vec{e_\theta} + r\sin\theta\begin{vmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{1}{r}\frac{\partial\widetilde{A}}{\partial\theta} & r\frac{1}{r\sin\theta}\left(-\frac{\partial\widetilde{A}}{\partial r}\right) \end{vmatrix}\vec{e_\phi}\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial b_\phi}{\partial\theta}\vec{e_r} - r\frac{\partial b_\phi}{\partial r}\vec{e_\theta} + r\sin\theta\left(\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\left(-\frac{\partial\widetilde{A}}{\partial r}\right)\right)\right) - r\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{1}{r}\frac{\partial\widetilde{A}}{\partial\theta}\right)\vec{e_\phi}\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial b_\phi}{\partial\theta}\vec{e_r} - r\frac{\partial b_\phi}{\partial r}\vec{e_\theta} - \left(r\frac{\partial}{\partial r}\left(\frac{\partial\widetilde{A}}{\partial r}\right) + \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\widetilde{A}}{\partial\theta}\right)\right)\vec{e_\phi}\right] \\
&= \frac{1}{r\sin\theta}\left[\frac{1}{r}\frac{\partial b_\phi}{\partial\theta}\vec{e_r} - \frac{\partial b_\phi}{\partial r}\vec{e_\theta} - \left(\frac{\partial}{\partial r}\left(\frac{\partial\widetilde{A}}{\partial r}\right) + \frac{1}{r^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\widetilde{A}}{\partial\theta}\right)\right)\vec{e_\phi}\right]
\end{aligned} \tag{2}$$

3. $\nabla \times \vec{B} = \alpha \vec{B}$

$$\frac{1}{r}\frac{\partial b_\phi}{\partial \theta} = \alpha \frac{1}{r}\frac{\partial}{\partial \theta}\widetilde{A} \tag{3}$$

$$\frac{\partial b_\phi}{\partial r} = \alpha \frac{\partial}{\partial r}\widetilde{A} \tag{4}$$

$$-\left(\frac{\partial}{\partial r}\left(\frac{\partial \widetilde{A}}{\partial r}\right) + \frac{1}{r^2}\sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \widetilde{A}}{\partial \theta}\right)\right) = \alpha \underbrace{\sin\theta\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)}_{b_\phi} \tag{5}$$

4. from (3), (4),

$$\frac{\partial b_\phi}{\partial \theta} = \alpha \frac{\partial \widetilde{A}}{\partial \theta}, \frac{\partial b_\phi}{\partial r} = \alpha \frac{\partial \widetilde{A}}{\partial r} \Rightarrow \frac{\frac{\partial b_\phi}{\partial \theta}}{\frac{\partial \widetilde{A}}{\partial \theta}} = \frac{\frac{\partial b_\phi}{\partial r}}{\frac{\partial \widetilde{A}}{\partial r}} \Rightarrow \frac{\partial b_\phi}{\partial \theta}\frac{\partial \widetilde{A}}{\partial r} - \frac{\partial b_\phi}{\partial r}\frac{\partial \widetilde{A}}{\partial \theta} = 0 = \begin{vmatrix} \frac{\partial b_\phi}{\partial r} & \frac{\partial b_\phi}{\partial \theta} \\ \frac{\partial \widetilde{A}}{\partial r} & \frac{\partial \widetilde{A}}{\partial \theta} \end{vmatrix} = J\left(b_\phi, \widetilde{A}\right) \tag{6}$$

$\therefore b_\phi = b_\phi\left(\widetilde{A}\right)$

5.

$$\begin{vmatrix} \frac{\partial b_\phi}{\partial r} & \frac{\partial b_\phi}{\partial \theta} \\ \frac{\partial \widetilde{A}}{\partial r} & \frac{\partial \widetilde{A}}{\partial \theta} \end{vmatrix} = 0 \Rightarrow \nabla b_\phi \times \nabla \widetilde{A} = 0, \nabla b_\phi = \begin{bmatrix} \frac{\partial b_\phi}{\partial r} \\ \frac{1}{r}\frac{\partial b_\phi}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial b_\phi}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial b_\phi}{\partial \widetilde{A}}\frac{\partial \widetilde{A}}{\partial r} \\ \frac{1}{r}\frac{\partial b_\phi}{\partial \widetilde{A}}\frac{\partial \widetilde{A}}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial b_\phi}{\partial \widetilde{A}}\frac{\partial \widetilde{A}}{\partial \phi} \end{bmatrix} = \frac{db_\phi}{d\widetilde{A}}\nabla \widetilde{A}. \tag{7}$$

$$\alpha \frac{\partial \widetilde{A}}{\partial r} = \frac{\partial b_\phi}{\partial r} = \frac{\partial b_\phi}{\partial \widetilde{A}}\frac{\partial \widetilde{A}}{\partial r} = \frac{db_\phi}{d\widetilde{A}}\frac{\partial \widetilde{A}}{\partial r} \Rightarrow \alpha = \frac{db_\phi}{d\widetilde{A}} \tag{8}$$

6. from (5)

$$-\frac{\partial^2 \widetilde{A}}{\partial r^2} - \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \widetilde{A}}{\partial \theta}\right) = \alpha b_\phi \tag{9}$$

where $\alpha = \frac{db_\phi}{d\widetilde{A}}$, so

$$-\frac{\partial^2 \widetilde{A}}{\partial r^2} - \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \widetilde{A}}{\partial \theta}\right) = b_\phi \frac{db_\phi}{d\widetilde{A}} = \frac{d}{d\widetilde{A}}\left(\frac{1}{2}b_\phi^2\right) \Rightarrow \frac{\partial^2 \widetilde{A}}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \widetilde{A}}{\partial \theta}\right) + \frac{d}{d\widetilde{A}}\left(\frac{1}{2}b_\phi^2\right) = 0 \tag{10}$$

Let $\widetilde{A} = \widetilde{A}(r,\theta) = F(\cos\theta)\, r^{-n}, b_\phi = a\widetilde{A}^{1+\frac{1}{n}}$,

$$\frac{\partial^2 \widetilde{A}}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \widetilde{A}}{\partial \theta}\right) + \frac{d}{d\widetilde{A}}\left(\frac{1}{2}b_\phi^2\right) = 0 \tag{11}$$

where $\widetilde{A} = \widetilde{A}(r,\theta) = F(\cos\theta)\, r^{-n}$, $b_\phi = a\widetilde{A}^{1+n^{-1}}$.

Then

$$\frac{\partial^2}{\partial r^2}\left(F(\cos\theta)\, r^{-n}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(F(\cos\theta)\, r^{-n}\right)\right] + \frac{d}{d\widetilde{A}}\left[\frac{1}{2}\left(a\widetilde{A}^{1+n^{-1}}\right)^2\right] = 0 \tag{12}$$

Now, firstly

$$\frac{\partial^2}{\partial r^2}\left(F(\cos\theta)\, r^{-n}\right) = \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(F(\cos\theta)\, r^{-n}\right)\right] = F(\cos\theta)\, n(n+1) r^{-(n+2)} \tag{13}$$

secondly,

$$\begin{aligned}
&\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(F(\cos\theta)\, r^{-n}\right)\right] \\
=&\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[r^{-n}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(F(\cos\theta)\right)\right] \\
=&r^{-(n+2)}\sin\theta\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\left(\frac{dF(\cos\theta)}{d\cos\theta}\right)\frac{d\cos\theta}{d\theta}\right] \\
=&r^{-(n+2)}\sin\theta\frac{\partial}{\partial\theta}\left[-\frac{1}{\sin\theta}\sin\theta\frac{dF(\cos\theta)}{d\cos\theta}\right] \\
=&-r^{-(n+2)}\sin\theta\frac{\partial}{\partial\theta}\left[\frac{dF(\cos\theta)}{d\cos\theta}\right] \\
=&-r^{-(n+2)}\sin\theta\frac{d}{d\theta}\left[\frac{dF(\cos\theta)}{d\cos\theta}\right] \\
=&-r^{-(n+2)}\sin\theta\frac{d}{-\frac{1}{\sin\theta}d\cos\theta}\left[\frac{dF(\cos\theta)}{d\cos\theta}\right] \quad \text{by using } d\theta = -\frac{1}{\sin\theta}d\cos\theta \\
=&\sin^2(\theta)\, r^{-(n+2)}\frac{d}{d\cos\theta}\left[\frac{dF(\cos\theta)}{d\cos\theta}\right] \\
=&\sin^2(\theta)\, r^{-(n+2)}\frac{d^2\left(F(\cos\theta)\right)}{d(\cos\theta)}
\end{aligned} \tag{14}$$

thirdly,

$$
\begin{aligned}
&\frac{d}{d\widetilde{A}}\left[\frac{1}{2}\left(a\widetilde{A}^{1+n^{-1}}\right)^2\right] \\
=&\frac{d}{d\widetilde{A}}\left[\frac{1}{2}a^2\widetilde{A}^{2+2n^{-1}}\right] \\
=&\frac{1}{2}a^2\left(2+2n^{-1}\right)\widetilde{A}^{1+2n^{-1}} \\
=&a^2\left(1+n^{-1}\right)\widetilde{A}^{1+2n^{-1}} \\
=&a^2\left(1+n^{-1}\right)\left[F(\cos\theta)\,r^{-n}\right]^{1+2n^{-1}} \\
=&a^2\left(1+n^{-1}\right)F^{1+2n^{-1}}(\cos\theta)\,r^{-(n+2)}
\end{aligned}
\tag{15}
$$

therefore,

$$
\begin{aligned}
&\frac{\partial^2}{\partial r^2}\left(F(\cos\theta)\,r^{-n}\right)+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(F(\cos\theta)\,r^{-n}\right)\right]+\frac{d}{d\widetilde{A}}\left[\frac{1}{2}\left(a\widetilde{A}^{1+n^{-1}}\right)^2\right] \\
=&F(\cos\theta)\,n(n+1)r^{-(n+2)}+\sin^2(\theta)\,r^{-(n+2)}\frac{d^2\left(F(\cos\theta)\right)}{d(\cos\theta)}+a^2\left(1+n^{-1}\right)F^{1+2n^{-1}}(\cos\theta)\,r^{-(n+2)} \\
=&r^{-(n+2)}\left[F(\cos\theta)\,n(n+1)r^{-(n+2)}+\sin^2(\theta)\,\frac{d^2\left(F(\cos\theta)\right)}{d(\cos\theta)}+a^2\left(1+n^{-1}\right)F^{1+2n^{-1}}(\cos\theta)\right] \\
=&0
\end{aligned}
\tag{16}
$$

One can get that

$$
F(\cos\theta)\,n(n+1)r^{-(n+2)}+\sin^2(\theta)\,\frac{d^2\left(F(\cos\theta)\right)}{d(\cos\theta)}+a^2\left(1+n^{-1}\right)F^{1+2n^{-1}}(\cos\theta)=0 \tag{17}
$$

Finally,

$$
\left(1-\mu^2\right)\frac{d^2F}{d\mu}+n(n+1)F+a^2\left(1+n^{-1}\right)F^{1+2n^{-1}}=\left(1-\mu^2\right)\frac{d^2F}{d\mu}+n(n+1)F+a^2\left(1+\frac{1}{n}\right)F^{1+\frac{2}{n}}=0 \tag{18}
$$

$$
\vec{B}=\frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial\widetilde{A}}{\partial\theta},-\frac{\partial\widetilde{A}}{\partial r},b_\phi\left(\widetilde{A}\right)\right) \tag{19}
$$

where $\widetilde{A}=\frac{F(\cos\theta)}{r}$, $b_\phi=a\frac{F^2(\cos\theta)}{r^2}$, $\cos\theta=\mu$.

Finally,

$$B_r = \frac{1}{r\sin\theta}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{F(\cos\theta)}{r}\right)\right] = \frac{1}{r\sin\theta}\frac{1}{r^2}\frac{\partial F}{\partial\cos\theta}\frac{\partial\cos\theta}{\partial\theta} = -\frac{1}{r\sin\theta}\frac{\sin\theta}{r^2}\frac{\partial F}{\partial\cos\theta} = \frac{1}{r^3}\frac{\partial F}{\partial\cos\theta} = \frac{1}{r^3}\frac{\partial F}{\partial\mu} \tag{20}$$

$$B_\theta - \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{F(\cos\theta)}{r}\right)\right] = \frac{1}{r\sin\theta}\frac{F}{r^2} = \frac{1}{r^3\sin\theta}F = \frac{1}{r^3}\frac{1}{(1-\cos^2\theta)^{\frac{1}{2}}}F = \frac{1}{r^3}\frac{1}{(1-\mu^2)^{\frac{1}{2}}}F, 0 \leqslant \theta < \pi, -1 \leqslant \mu < 1 \tag{21}$$

$$B_\phi = a\frac{1}{r\sin\theta}\frac{F^2(\cos\theta)}{r^2} = a\frac{1}{r^3\sin\theta}F^2 = a\frac{1}{r^3}\frac{1}{(1-\mu^2)^{\frac{1}{2}}}F^2 \tag{22}$$

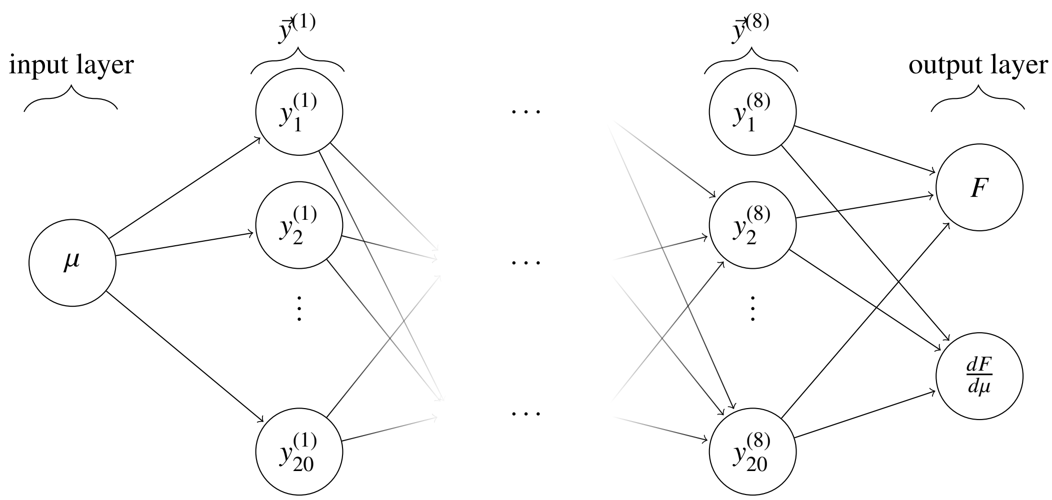

Fig. 1: Our Module

1. $i = 1$

$$\vec{y}^{(1)} = f\left(W^{(1)}x + b^{(1)}\right) \in \mathbb{R}^{20\times1} \tag{23}$$

where $x \in \mathbb{R}^{1\times1}, W^{(1)} \in \mathbb{R}^{20\times1}, b^{(1)} \in \mathbb{R}^{20\times1}$, $f = tanh(\cdot)$.

2. $2 \leq i \leq 8$

$$\vec{x}^{(i+1)} = \vec{y}^{(i)}, \quad 1 \leq i \in \mathbb{Z} \leq 8 \tag{24}$$

$$\vec{y}^{(i)} = f\left(W^{(i)}x^{(i)} + \vec{b}^{(i)}\right) = x^{(i+1)} \in \mathbb{R}^{20\times1}, \quad 1 \leq i \in \mathbb{Z} \leq 8 \tag{25}$$

where $\vec{x}^{(i)} = \in \mathbb{R}^{20\times1}, W^{(i)} \in \mathbb{R}^{20\times20}, \vec{b}^{(i)} \in \mathbb{R}^{20\times1}$.

3. $i = 9$

$$\vec{y}^{(9)} = f\left(W^{(9)}\vec{x}^{(9)} + \vec{b}^{(9)}\right) = \begin{bmatrix} F \\ F' \end{bmatrix} \in \mathbb{R}^{2\times 1} \tag{26}$$

where $\vec{x}^{(9)} =\in \mathbb{R}^{20\times 1}$, $W^{(9)} \in \mathbb{R}^{2\times 20}$, $\vec{b}^{(9)} \in \mathbb{R}^{2\times 1}$

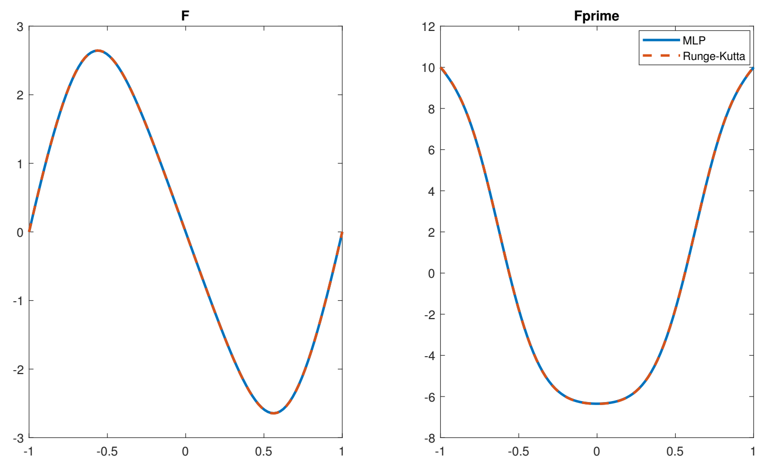

Fig. 2: Numerical solution to (18) when $n = 1$.

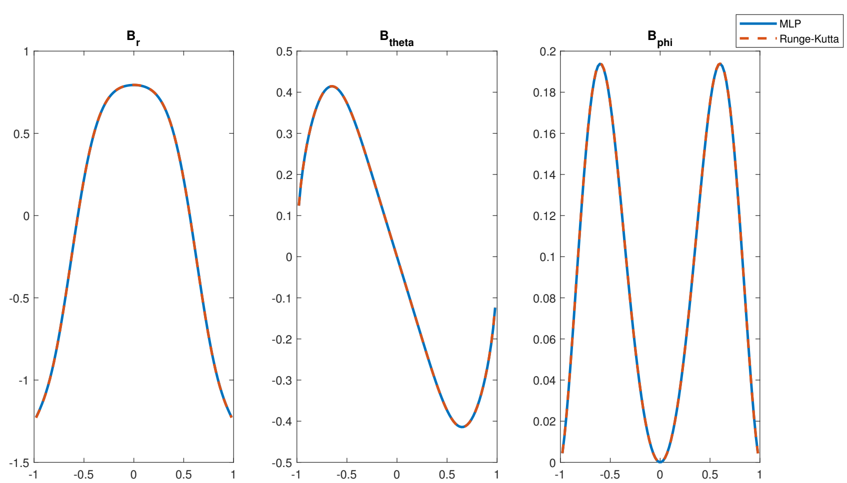

Fig. 3: Numerical solution to (19).

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