# Subspace Clustering via Kernel Method

## 1 Preliminary

In this paper, $x, y \in \mathbb{R}^{D\times 1}, Y = [y_1, y_2, \cdots, y_N] \in \mathbb{R}^{D\times N}$ and $C \in \mathbb{R}^{N\times N}$. $\langle x, y\rangle$ is the inner product of $x$ and $y$, $\|x - y\|^2 = \langle x - y, x - y\rangle$ .

**Definition 1.** *Let $\Phi : \mathbb{R}^N \to \mathcal{H}$ be a mapping from the input space to the reproducing kernel Hilbert space $\mathcal{H}$. Let $\mathcal{K}_{YY} \in \mathbb{R}^{N\times N}$ be a positive semidefinite kernel Gram matrix whose elements are computed as:*

$$[\mathcal{K}_{YY}]_{ij} = [\langle \Phi(Y), \Phi(Y)\rangle_{\mathcal{H}}]_{ij} = \Phi(y_i)^T \Phi(y_j) = \kappa(y_i, y_j) \tag{1}$$

*where $\kappa : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$ is the kernel function and*

$$\Phi(Y) = [\Phi(y_1), \Phi(y_2), \cdots, \Phi(y_N)] \tag{2}$$

*and*

$$\mathcal{K}_{YY} = \begin{bmatrix} \kappa(y_1, y_1) & \kappa(y_1, y_2) & \cdots & \kappa(y_1, y_N) \\ \kappa(y_2, y_1) & \kappa(y_2, y_2) & \cdots & \kappa(y_2, y_N) \\ \vdots & \vdots & \ddots & \vdots \\ \kappa(y_N, y_1) & \kappa(y_N, y_2) & \cdots & \kappa(y_N, y_N) \end{bmatrix} = \Phi(Y)^T \Phi(Y) \tag{3}$$

*where $\mathcal{K}_{YY} \in \mathbb{R}^{N\times N}$.*

**Remark 1.** *Some commonly used kernels such as:*

1. *polynomial kernel:*

$$\kappa(x, y) = (\langle x, y\rangle + a)^b \tag{4}$$

*where $a, b$ are the parameter of the kernel function.*

2. *Gaussian kernel*

$$\kappa(x, y) = e^{-\sigma\|x-y\|^2} \tag{5}$$

*where $\sigma$ is the parameter of the kernel function.*

$\mathcal{K}_{YY}$ is symmetric matrices if we select the kernels as above. In our model, the kernels are adopted as remark 1.

## 2 Related Work

In paper [1], they give KSSC model as:

$$\begin{aligned} &\min_C \|C\|_1 + \lambda \|\Phi(Y) - \Phi(Y) C\|_F^2 \\ &s.t.\ diag(C) = 0,\ C^T 1 = 1 \end{aligned} \tag{6}$$

**Remark 2.**

$$
\begin{aligned}
&\|\Phi(Y)-\Phi(Y)C\|_F^2\\
=&\langle \Phi(Y)-\Phi(Y)C,\Phi(Y)-\Phi(Y)C\rangle\\
=&\langle \Phi(Y),\Phi(Y)\rangle-2\langle \Phi(Y),\Phi(Y)C\rangle+\langle \Phi(Y)C,\Phi(Y)C\rangle\\
=&tr\left(\Phi(Y)^T\Phi(Y)\right)-2tr\left(\Phi(Y)^T\Phi(Y)C\right)+tr\left(C^T\Phi(Y)^T\Phi(Y)C\right)\\
=&tr(\mathcal{K}_{YY})-2tr(\mathcal{K}_{YY}C)+tr\left(C^T\mathcal{K}_{YY}C\right)
\end{aligned}
\tag{7}
$$

# 3 Our Model

Our model KSLRSC:

$$
\begin{gathered}
\min_C \alpha\|C\|_{l_p}^p+\beta\|C\|_{S_q}^q+\gamma\|\Phi(Y)-\Phi(Y)C\|_F^2\\
s.t.\ diag(C)=0, C^T1=1
\end{gathered}
\tag{8}
$$

or

$$
\begin{gathered}
\min_C \alpha\|C\|_{l_p}^p+\beta\|C\|_{S_q}^q+\gamma\|E\|_{l_{2r}}^r\\
s.t.\ diag(C)=0, E=\Phi(Y)-\Phi(Y)C, C^T1=1
\end{gathered}
\tag{9}
$$

**Remark 3.** $\|E\|_{l_{2r}}^r$ *is similar to* $\|E\|_{21}$*, we will give the definition of* $\|E\|_{2,2/3}^{2/3}$ *and* $\|E\|_{2,1/3}^{1/3}$*, maybe other form of* $E$ *in the proceeding.*

# 4 Optimization for KSSC

To solve problem (6) via ADMM, we need an auxiliary $A \in \mathbb{R}$,

$$
\begin{gathered}
\min \|C\|_1+\lambda tr\left(K_{YY}-2K_{YY}A+A^TK_{YY}A\right)\\
s.t.\ \ A=C-diag(C), A^T1=1
\end{gathered}
\tag{10}
$$

the Lagrange formulation of (10) is:

$$
\begin{aligned}
\mathcal{L}(C,A,Y_1,Y_2)=&\|C\|_1+\lambda tr\left(\mathcal{K}_{YY}-2\mathcal{K}_{YY}A+A^T\mathcal{K}_{YY}A\right)\quad+\\
&\langle Y_1, A-C+diag(C)\rangle+\left\langle Y_2, A^T1-1\right\rangle\quad+\\
&\frac{\mu}{2}\left(\|A-C+diag(C)\|_F^2+\left\|A^T1-1\right\|_F^2\right)
\end{aligned}
\tag{11}
$$

where $A, Y_1 \in \mathbb{R}^{N\times N}, Y_2, 1 \in \mathbb{R}^{N\times 1}$, and entries of 1 are all 1s.

Update one variable when fix others:

1. update A

$$
\begin{aligned}
A^* &= \arg\min_{A} \lambda tr\left(\mathcal{K}_{YY} - 2\mathcal{K}_{YY}A + A^T\mathcal{K}_{YY}A\right) + \langle Y_1, A - C + diag\left(C\right)\rangle + \\
&\quad \left\langle Y_2, A^T 1 - 1\right\rangle + \frac{\mu}{2}\left(\left\|A - C + diag\left(C\right)\right\|_F^2 + \left\|A^T 1 - 1\right\|_F^2\right) \\
&= \arg\min_{A} \lambda tr\left(A^T\mathcal{K}_{YY}A - 2\mathcal{K}_{YY}A\right) \quad + \\
&\quad \frac{\mu}{2}\left(\left\|A - C + diag\left(C\right) + \frac{Y_1}{\mu}\right\|_F^2 + \left\|A^T 1 - 1 + \frac{Y_2}{\mu}\right\|_F^2\right)
\end{aligned} \tag{12}
$$

Let

$$
J = \lambda tr\left(A^T\mathcal{K}_{YY}A - 2\mathcal{K}_{YY}A\right) + \frac{\mu}{2}\left(\left\|A - C + diag\left(C\right) + \frac{Y_1}{\mu}\right\|_F^2 + \left\|A^T 1 - 1 + \frac{Y_2}{\mu}\right\|_F^2\right) \tag{13}
$$

then

$$
\frac{\partial J}{\partial A} = 0 \tag{14}
$$

i.e.

$$
\begin{aligned}
&\frac{\partial J}{\partial A} = 0 \\
&= 2\lambda\left(\mathcal{K}_{YY}A - \mathcal{K}_{YY}\right) + \mu\left[\left(A - C + diag\left(C\right) + \frac{Y_1}{\mu}\right) + 1\left(1^T A - 1^T + \frac{Y_2^T}{\mu}\right)\right]
\end{aligned} \tag{15}
$$

then we can get

$$
\left(2\lambda\mathcal{K}_{YY} + \mu I + 11^T\right) A = 2\lambda\mathcal{K}_{YY} + \mu\left(C - diag\left(C\right) + 11^T\right) - Y_1 - 1Y_2^T \tag{16}
$$

so, we can get that

$$
A = \left(2\lambda\mathcal{K}_{YY} + \mu I + 11^T\right)^{-1}\left[2\lambda\mathcal{K}_{YY} + \mu\left(C - diag\left(C\right) + 11^T\right) - Y_1 - 1Y_2^T\right] \tag{17}
$$

2. update C

$$
\begin{aligned}
C^* &= \arg\min_{C} \|C\|_1 + \langle Y_1, A - C + diag\left(C\right)\rangle + \frac{\mu}{2}\left\|A - C + diag\left(C\right)\right\|_F^2 \\
&= \arg\min_{C} \|C\|_1 + \frac{\mu}{2}\left\|A - C + diag\left(C\right) + \frac{Y_1}{\mu}\right\|_F^2 \\
&= \arg\min_{C} \|C\|_1 + \frac{\mu}{2}\left\|C - \left(A + \frac{Y_1}{\mu}\right)\right\|_F^2 \\
&= \arg\min_{C} \frac{1}{\mu}\|C\|_1 + \frac{1}{2}\left\|C - \left(A + \frac{Y_1}{\mu}\right)\right\|_F^2
\end{aligned} \tag{18}
$$

3. update $Y_1$

$$
Y_1 \leftarrow Y_1 + \mu\left(A - C + diag\left(C\right)\right) \tag{19}
$$

4. update $Y_2$

$$
Y_2 \leftarrow Y_2 + \mu\left(A^T 1 - 1\right) \tag{20}
$$

5. update $\mu$

$$
\mu \leftarrow \min\left(\rho\mu, \mu_{\max}\right) \tag{21}
$$

## 5 Optimization for Our Model KSLRSC

our model is:

$$\begin{aligned} \min_{C,D,F,E} \alpha \|C\|_{l_p}^p + \beta \|D\|_{S_p}^p + \gamma \|\Phi(Y) - \Phi(Y) F\|_F^2 \\ s.t.\ F = C - diag(C), F = D, F^T 1 = 1 \end{aligned} \tag{22}$$

### 5.1 Sub-Model

1. $p = 1$

$$\begin{aligned} \min\ \alpha \|C\|_1 + \beta \left[\frac{1}{2}\left(\|U\|_F^2 + \|V\|_F^2\right)\right] + \gamma \|\Phi(Y) - \Phi(Y) F\|_F^2 \\ s.t.\ F = C - diagC, F = UV^T, F^T 1 = 1 \end{aligned} \tag{23}$$

2. $p = 2/3$

$$\begin{aligned} \min\ \alpha \|C\|_{l_{2/3}}^{2/3} + \beta \left[\frac{2}{3}\|M\|_* + \frac{1}{3}\|V\|_F^2\right] + \gamma \|\Phi(Y) - \Phi(Y) F\|_F^2 \\ s.t.\ F = C - diagC, F = UV^T, F^T 1 = 1, M = U \end{aligned} \tag{24}$$

3. $p = 1/2$

$$\begin{aligned} \min\ \alpha \|C\|_{l_{1/2}}^{1/2} + \beta \left[\frac{1}{2}\left(\|M\|_* + \|N\|_*\right)\right] + \gamma \|\Phi(Y) - \Phi(Y) F\|_F^2 \\ s.t.\ F = C - diagC, F = UV^T, F^T 1 = 1, M = U, N = V \end{aligned} \tag{25}$$

### 5.2 Sub-ALM

ALMs are responding for 23,24,25 as flowing respectively.

1. $p = 1$

$$\begin{aligned} \mathcal{L}(U,V,C,F) = \alpha\|C\|_1 + \frac{\beta}{2}\left(\|U\|_F^2 + \|V\|_F^2\right) + \gamma \|\Phi(Y) - \Phi(Y)\|_F^2 + \\ \langle Y_1, F - C + diagC\rangle + \left\langle Y_2, F - UV^T\right\rangle + \left\langle Y_3, F^T 1 - 1\right\rangle + \\ \frac{\mu}{2}\left(\|F - C + diagC\|_F^2 + \left\|F - UV^T\right\|_F^2 + \left\|F^T 1 - 1\right\|_F^2\right) \end{aligned} \tag{26}$$

2. $p = 2/3$

$$\begin{aligned} \mathcal{L}(U,V,M,C,F) = \alpha \|C\|_{l_{2/3}}^{2/3} + \frac{\beta}{3}\left(2\|M\|_* + \|V\|_F^2\right) + \gamma \|\Phi(Y) - \Phi(Y)\|_F^2 + \\ \langle Y_1, F - C + diagC\rangle + \left\langle Y_2, F - UV^T\right\rangle + \left\langle Y_3, F^T 1 - 1\right\rangle + \langle Y_4, M - U\rangle + \\ \frac{\mu}{2}\left(\|F - C + diagC\|_F^2 + \left\|F - UV^T\right\|_F^2 + \left\|F^T 1 - 1\right\|_F^2 + \|M - U\|_F^2\right) \end{aligned} \tag{27}$$

3. $p = 1/2$

$$\begin{aligned}
&\mathcal{L}(U,V,M,N,C,F) = \alpha \|C\|_{l_{1/2}}^{1/2} + \frac{\beta}{2}\left(\|M\|_* + \|N\|_*\right) + \gamma \|\Phi(Y) - \Phi(Y)\|_F^2 + \\
&\langle Y_1, F - C + diagC \rangle + \left\langle Y_2, F - UV^T \right\rangle + \left\langle Y_3, F^T 1 - 1 \right\rangle + \\
&\qquad \langle Y_4, M - U \rangle + \langle Y_5, N - V \rangle + \\
&\frac{\mu}{2}\left(\|F - C + diagC\|_F^2 + \left\|F - UV^T\right\|_F^2 + \left\|F^T 1 - 1\right\|_F^2 + \|M - U\|_F^2 + \|N - V\|_F^2\right)
\end{aligned} \tag{28}$$

### 5.3 Sub-ADMM

- $p = 1$

1. update U

$$\begin{aligned}
U^* &= \arg\min_U \frac{\beta}{2}\|U\|_F^2 + \left\langle Y_2, F - UV^T \right\rangle + \frac{\mu}{2}\left\|F - UV^T\right\|_F^2 \\
&== \arg\min_U \frac{\beta}{2}\|U\|_F^2 + \frac{\mu}{2}\left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2
\end{aligned} \tag{29}$$

Let

$$J = \frac{\beta}{2}\|U\|_F^2 + \frac{\mu}{2}\left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2 \tag{30}$$

then

$$\frac{\partial J}{\partial U} = 0 \Rightarrow \beta U - \mu\left(F - UV^T + \frac{Y_2}{\mu}\right)V = 0 \Leftrightarrow U\left(\beta I + \mu V^T V\right) = \mu F V + Y_2 V \tag{31}$$

finally, we can get

$$U = \left(\mu F V + Y_2 V\right)\left(\beta I + \mu V^T V\right)^{-1} \tag{32}$$

2. update V

$$\begin{aligned}
V^* &= \arg\min_V \frac{\beta}{2}\|V\|_F^2 + \left\langle Y_2, F - UV^T \right\rangle + \frac{\mu}{2}\left\|F - UV^T\right\|_F^2 \\
&= \arg\min_V \frac{\beta}{2}\|V\|_F^2 + \frac{\mu}{2}\left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2 \\
&= \arg\min_V \frac{\beta}{2}\|V\|_F^2 + \frac{\mu}{2}\left\|F^T - VU^T + \frac{Y_2^T}{\mu}\right\|_F^2
\end{aligned} \tag{33}$$

let

$$J = \frac{\beta}{2}\|V\|_F^2 + \frac{\mu}{2}\left\|F^T - VU^T + \frac{Y_2^T}{\mu}\right\|_F^2 \tag{34}$$

then

$$\begin{aligned}
&\frac{\partial J}{\partial V} = 0 \\
&\quad \Rightarrow \beta V - \mu\left(F^T - VU^T + \frac{Y_2^T}{\mu}\right)U = 0 \Leftrightarrow V\left(\beta I + \mu U^T U\right) = \mu F^T U + Y_2^T U
\end{aligned} \tag{35}$$

finally, we can get

$$V = \left(\mu F^T U + Y_2^T U\right)\left(\beta I + \mu U^T U\right)^{-1} \tag{36}$$

3. update C

$$
\begin{aligned}
C^* &= \arg\min_C \alpha\|C\|_1 + \langle Y_1, F - C + diagC\rangle + \frac{\mu}{2}\|F - C + diagC\|_F^2 \\
&= \arg\min_C \alpha\|C\|_1 + \frac{\mu}{2}\left\|C - F - \frac{Y_1}{\mu}\right\|_F^2 \\
&= \arg\min_C \frac{\alpha}{\mu}\|C\|_1 + \frac{1}{2}\left\|C - F - \frac{Y_1}{\mu}\right\|_F^2
\end{aligned} \tag{37}
$$

then

$$
C \leftarrow C - diagC \tag{38}
$$

- $p = 2/3$

1. update U

$$
\begin{aligned}
U^* &= \arg\min_U \left\langle Y_2, F - UV^T\right\rangle + \langle Y_4, M - U\rangle + \frac{\mu}{2}\left(\left\|F - UV^T\right\|_F^2 + \|M - U\|_F^2\right) \\
&= \arg\min_U \frac{\mu}{2}\left(\left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2 + \left\|M - U + \frac{Y_4}{\mu}\right\|_F^2\right) \\
&= \arg\min_U \left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2 + \left\|M - U + \frac{Y_4}{\mu}\right\|_F^2
\end{aligned} \tag{39}
$$

Let

$$
J = \left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2 + \left\|M - U + \frac{Y_4}{\mu}\right\|_F^2 \tag{40}
$$

then

$$
\begin{aligned}
\frac{\partial J}{\partial U} = 0 &\Rightarrow -\left(F - UV^T + \frac{Y_2}{\mu}\right)V - \left(M - U + \frac{Y_4}{\mu}\right) = 0 \\
&\Leftrightarrow U\left(V^TV + I\right) = FV + M + \frac{Y_2V + Y_4}{\mu}
\end{aligned} \tag{41}
$$

finally, we can get

$$
U = \left(FV + M + \frac{Y_2V + Y_4}{\mu}\right)\left(V^TV + I\right)^{-1} \tag{42}
$$

2. update V

$$
\begin{aligned}
V^* &= \arg\min_V \frac{\beta}{3}\|V\|_F^2 + \left\langle Y_2, F - UV^T\right\rangle + \frac{\mu}{2}\left\|F - UV^T\right\|_F^2 \\
&= \arg\min_V \frac{\beta}{3}\|V\|_F^2 + \frac{\mu}{2}\left\|F^T - VU^T + \frac{Y_2^T}{\mu}\right\|_F^2
\end{aligned} \tag{43}
$$

Let

$$
J = \frac{\beta}{3}\|V\|_F^2 + \frac{\mu}{2}\left\|F^T - VU^T + \frac{Y_2^T}{\mu}\right\|_F^2 \tag{44}
$$

then

$$\begin{aligned}\frac{\partial J}{\partial V} = 0 \Rightarrow \frac{2}{3}\beta V - \mu\left(F^T - VU^T + \frac{Y_2^T}{\mu}\right)U = 0\\ \Leftrightarrow V\left(\frac{2}{3}\beta I + \mu U^T U\right) = \mu F^T U + Y_2^T U\end{aligned} \tag{45}$$

finally, we can get

$$V = \left(\mu F^T U + Y_2^T U\right)\left(\frac{2}{3}\beta I + \mu U^T U\right)^{-1} \tag{46}$$

3. update M

$$\begin{aligned}M^* &= \arg\min_M \frac{2}{3}\beta\|M\|_* + \langle Y_4, M - U\rangle + \frac{\mu}{2}\|M-U\|_F^2\\ &= \arg\min_M \frac{2}{3}\beta\|M\|_* + \frac{\mu}{2}\left\|M - U + \frac{Y_4}{\mu}\right\|_F^2\\ &= \arg\min_M \frac{2\beta}{3\mu}\|M\|_* + \frac{1}{2}\left\|M - U + \frac{Y_4}{\mu}\right\|_F^2\end{aligned} \tag{47}$$

4. update C

$$\begin{aligned}C^* &= \arg\min_C \alpha\|C\|_{l_{2/3}}^{2/3} + \langle Y_1, F - C + diagC\rangle + \frac{\mu}{2}\|F - C + diagC\|_F^2\\ &= \arg\min_C \alpha\|C\|_{l_{2/3}}^{2/3} + \frac{\mu}{2}\left\|C - \left(F + \frac{Y_1}{\mu}\right)\right\|_F^2\\ &= \arg\min_C \frac{2\alpha}{\mu}\|C\|_{l_{2/3}}^{2/3} + \left\|C - \left(F + \frac{Y_1}{\mu}\right)\right\|_F^2\end{aligned} \tag{48}$$

then

$$C \leftarrow C - diagC \tag{49}$$

- $p = 1/2$

1. update U

$$\begin{aligned}U^* &= \arg\min_U \left\langle Y_2, F - UV^T\right\rangle + \langle Y_4, M - U\rangle + \frac{\mu}{2}\left(\left\|F - UV^T\right\|_F^2 + \|M - U\|_F^2\right)\\ &= \arg\min_U \left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2 + \left\|M - U + \frac{Y_4}{\mu}\right\|_F^2\end{aligned} \tag{50}$$

let

$$J = \left\|F - UV^T + \frac{Y_2}{\mu}\right\|_F^2 + \left\|M - U + \frac{Y_4}{\mu}\right\|_F^2 \tag{51}$$

then

$$\begin{aligned}\frac{\partial J}{\partial U} = 0 \Rightarrow -\left(F - UV^T + \frac{Y_2}{\mu}\right)V - \left(M - U + \frac{Y_4}{\mu}\right) = 0\\ \Leftrightarrow U\left(V^T V + I\right) = FV + M + \frac{Y_2 V + Y_4}{\mu}\end{aligned} \tag{52}$$

finally, we can get

$$U = \left(FV + M + \frac{Y_2 V + Y_4}{\mu}\right)\left(V^T V + I\right)^{-1} \tag{53}$$

2. update V

$$
\begin{aligned}
V^* &= \arg\min_V \left\langle Y_2, F - UV^T \right\rangle + \langle Y_5, N - V \rangle + \frac{\mu}{2}\left( \left\| F - UV^T \right\|_F^2 + \| N - V \|_F^2 \right) \\
&= \arg\min_V \left\| F - UV^T + \frac{Y_2}{\mu} \right\|_F^2 + \left\| N - V + \frac{Y_5}{\mu} \right\|_F^2 \\
&= \arg\min_V \left\| F^T - VU^T + \frac{Y_2^T}{\mu} \right\|_F^2 + \left\| N - V + \frac{Y_5}{\mu} \right\|_F^2
\end{aligned} \tag{54}
$$

let

$$
J = \left\| F^T - VU^T + \frac{Y_2^T}{\mu} \right\|_F^2 + \left\| N - V + \frac{Y_5}{\mu} \right\|_F^2 \tag{55}
$$

then

$$
\begin{aligned}
\frac{\partial J}{\partial V} = 0 \Rightarrow - \left( F^T - VU^T + \frac{Y_2^T}{\mu} \right) U - \left( N - V + \frac{Y_5}{\mu} \right) \\
\Leftrightarrow V \left( U^T U + I \right) = F^T U + N + \frac{Y_2^T U + Y_5}{\mu}
\end{aligned} \tag{56}
$$

finally, we can get

$$
V = \left( F^T U + N + \frac{Y_2^T U + Y_5}{\mu} \right) \left( U^T U + I \right)^{-1} \tag{57}
$$

3. update M

$$
\begin{aligned}
M^* &= \arg\min_M \frac{\beta}{2} \| M \|_* + \langle Y_4, M - U \rangle + \frac{\mu}{2} \| M - U \|_F^2 \\
&= \arg\min_M \frac{\beta}{2} \| M \|_* + \frac{\mu}{2} \left\| M - U + \frac{Y_4}{\mu} \right\|_F^2 \\
&= \arg\min_M \frac{\beta}{2\mu} \| M \|_* + \frac{1}{2} \left\| M - U + \frac{Y_4}{\mu} \right\|_F^2
\end{aligned} \tag{58}
$$

4. update N

$$
N^* = \arg\min_N \frac{\beta}{2\mu} \| N \|_* + \frac{1}{2} \left\| N - V + \frac{Y_5}{\mu} \right\|_F^2 \tag{59}
$$

5. update C

$$
\begin{aligned}
C^* &= \arg\min_C \alpha \| C \|_{l_{1/2}}^{1/2} + \langle Y_1, F - C + diagC \rangle + \frac{\mu}{2} \| F - C + diagC \|_F^2 \\
&= \arg\min_C \frac{2\alpha}{\mu} \| C \|_{l_{1/2}}^{1/2} + \left\| C - \left( F + \frac{Y_1}{\mu} \right) \right\|_F^2
\end{aligned} \tag{60}
$$

$$
C \leftarrow C - diagC \tag{61}
$$

we can get that

$$
\begin{aligned}
F^* =& \arg\min_F \gamma \left\| \Phi(Y) - \Phi(Y) F \right\|_F^2 + \langle Y_1, F - C + diagC \rangle + \left\langle Y_2, F - UV^T \right\rangle + \\
& \left\langle Y_3, F^T 1 - 1 \right\rangle \quad + \\
& \frac{\mu}{2} \left( \| F - C + diagC \|_F^2 + \left\| F - UV^T \right\|_F^2 + \left\| F^T 1 - 1 \right\|_F^2 \right) \\
=& \arg\min_F \gamma \left\| \Phi(Y) - \Phi(Y) F \right\|_F^2 + \\
& \frac{\mu}{2} \left( \left\| F - C + diagC + \frac{Y_1}{\mu} \right\|_F^2 + \left\| F - UV^T + \frac{Y_2}{\mu} \right\|_F^2 + \left\| F^T 1 - 1 + \frac{Y_3}{\mu} \right\|_F^2 \right) \\
=& \arg\min_F \gamma \left\| \Phi(Y) - \Phi(Y) F \right\|_F^2 + \\
& \frac{\mu}{2} \left( \left\| F - C + diagC + \frac{Y_1}{\mu} \right\|_F^2 + \left\| F - UV^T + \frac{Y_2}{\mu} \right\|_F^2 + \left\| 1^T F - 1^T + \frac{Y_3^T}{\mu} \right\|_F^2 \right) \\
\overset{by\ Remark\ 2}{=}& \arg\min_F \gamma\ Tr \left( \mathcal{K}_{YY} - 2\mathcal{K}_{YY} F + F^T \mathcal{K}_{YY} F \right) \\
& \frac{\mu}{2} \left( \left\| F - C + diagC + \frac{Y_1}{\mu} \right\|_F^2 + \left\| F - UV^T + \frac{Y_2}{\mu} \right\|_F^2 + \left\| 1^T F - 1^T + \frac{Y_3^T}{\mu} \right\|_F^2 \right)
\end{aligned} \tag{62}
$$

Whenever $p = 1, 2/3, 1/2$ . let

$$
\begin{aligned}
J =& \gamma\ Tr \left( \mathcal{K}_{YY} - 2\mathcal{K}_{YY} F + F^T \mathcal{K}_{YY} F \right) + \\
& \frac{\mu}{2} \left( \left\| F - C + diagC + \frac{Y_1}{\mu} \right\|_F^2 + \left\| F - UV^T + \frac{Y_2}{\mu} \right\|_F^2 + \left\| 1^T F - 1^T + \frac{Y_3^T}{\mu} \right\|_F^2 \right)
\end{aligned} \tag{63}
$$

then

$$
\begin{aligned}
\frac{\partial J}{\partial F} = 0 \Rightarrow & \gamma \left( -2\mathcal{K}_{YY} + 2\mathcal{K}_{YY} F \right) + \\
& \mu \left[ \left( F - C + diagC + \frac{Y_1}{\mu} \right) + \left( F - UV^T + \frac{Y_2}{\mu} \right) + 1 \left( 1^T F - 1^T + \frac{Y_3^T}{\mu} \right) \right] = 0
\end{aligned} \tag{64}
$$

then

$$
\left( 2\gamma \mathcal{K}_{YY} + 2\mu I + \mu 1 1^T \right) F = 2\gamma \mathcal{K}_{YY} + \mu \left( C - diagC + UV^T + 1 1^T \right) + \left( Y_1 + Y_2 + 1 Y_3^T \right) \tag{65}
$$

finally, we can get

$$
\begin{aligned}
& F = \\
& \left( 2\gamma \mathcal{K}_{YY} + 2\mu I + \mu 1 1^T \right)^{-1} \left[ 2\gamma \mathcal{K}_{YY} + \mu \left( C - diagC + UV^T + 1 1^T \right) + \left( Y_1 + Y_2 + 1 Y_3^T \right) \right]
\end{aligned} \tag{66}
$$

# 6 Appendix

1. Self-Expressiveness

$$
X = (x_1, x_2, \cdots, x_n), X = XC \tag{67}
$$

where $x_i \in \mathbb{R}^{m\times 1}, C \in \mathbb{R}^{n\times n}$.

Especially

$$x_i = (x_1, x_2, \cdots, x_n)\begin{pmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{pmatrix} \tag{68}$$

if

$$\sum_{j=1}^{n} c_{ji} = (c_{1i}, c_{2i}, \cdots, c_{ni})\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \tag{69}$$

where 1 is a scalar. Then it is a affine combination.

[1] V. Patel and R. Vidal , “Kernel Sparse Subspace Clustering,” *IEEE International Conference on Image Processing* , 2014, page: 2849–2853.

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