A proposition of nuclear norm

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Notation 1. *Let*

$$X = LR^T = U\Sigma V^T \in \mathbb{R}^{m\times n}$$

Notation 2.

$$L \in \mathbb{R}^{m\times k}, R \in \mathbb{R}^{n\times k}, U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}, \Sigma \in \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Notation 3.

$$\|X\|_* = \sum_{i=1}^{r} \sigma_i$$

Proposition 1.

$$\|X\|_* = \min \|L\|_F \cdot \|R\|_F = \min \frac{1}{2}\left(\|L\|_F^2 + \|R\|_F^2\right) \tag{1}$$

Proof. We only to prove that

$$\underbrace{\min \|L\|_F \cdot \|R\|_F}_{1} \leqslant \underbrace{\min \frac{1}{2}\left(\|L\|_F^2 + \|R\|_F^2\right)}_{2} \leqslant \underbrace{\|X\|_*}_{3} \leqslant \underbrace{\min \|L\|_F \cdot \|R\|_F}_{1} \tag{2}$$

1. "1" $\leq$ "2"

$$\min \frac{1}{2}\left(\|L\|_F^2 + \|R\|_F^2\right) = \frac{1}{2}\left(\|L_0\|_F^2 + \|R_0\|_F^2\right) \geqslant \|L_0\|_F \cdot \|R_0\|_F \geqslant \min \|L\|_F \cdot \|R\|_F \tag{3}$$

2. "2" $\leq$ "3"

$$\text{Let} \quad L = U(\Sigma)^{\frac{1}{2}}, R = V(\Sigma)^{\frac{1}{2}} \Rightarrow \|L\|_F^2 = \|X\|_* = \|R\|_F^2 \tag{4}$$

so,

$$\min \frac{\|L\|_F^2 + \|R\|_F^2}{2} \leqslant \|X\|_* \tag{5}$$

3. "3" $\leq$ "1"

$$
\begin{aligned}
\|X\|_* = \sum_{i=1}^{r} \sigma_i &= \sum_{i=1}^{r} {u_i}^T u_i \Sigma {v_i}^T v_i \\
&= \underbrace{{u_1}^T u_1 \Sigma {v_1}^T v_1 + \underbrace{{u_1}^T u_2 \Sigma {v_2}^T v_1}_{0} + \cdots + \underbrace{{u_1}^T u_r \Sigma {v_r}^T v_1}_{0}}_{u_1^T A v_1} \\
&\quad + \underbrace{\underbrace{{u_2}^T u_1 \Sigma {v_1}^T v_2}_{0} + {u_2}^T u_2 \Sigma {v_2}^T v_2 + \cdots + \underbrace{{u_2}^T u_r \Sigma {v_r}^T v_2}_{0}}_{u_2^T A v_2} \\
&\quad + \cdots \\
&\quad + \underbrace{\underbrace{{u_r}^T u_1 \Sigma {v_1}^T v_r}_{0} + \underbrace{{u_r}^T u_2 \Sigma {v_2}^T v_r}_{0} + \cdots + {u_r}^T u_r \Sigma {v_r}^T v_r}_{u_r^T A v_r} \\
&= \sum_{i=1}^{r} u_i^T A v_i \\
&= \sum_{i=1}^{r} \left\langle u_i^T L, v_i^T R^T \right\rangle
\end{aligned} \tag{6}
$$

so,

$$
\|A\|_* \leqslant \sup \sum_{i=1}^{r} \left\langle O_{1_i} L, O_{2_i} R^T \right\rangle \quad \text{for } \left\{O_{1_i}\right\}, \left\{O_{2_i}\right\} \text{orthonormal basis} \tag{7}
$$

and we can get that

$$
\begin{aligned}
\|A\|_* &\leqslant \sup \sum_{i=1}^{r} \left\langle O_{1_i} L, O_{2_i} R^T \right\rangle \\
&\leqslant \sup \sum \|O_{1_i} L\|_F \sum \left\|O_{2_i} R^T\right\|_F \\
\text{apply} &\ \text{ Cauchy–Schwarz inequality} \\
&\leqslant \sup \left(\sum \|O_{1_i} L\|_F^2\right)^{\frac{1}{2}} \left(\sum \left\|O_{2_i} R^T\right\|_F^2\right)^{\frac{1}{2}} \\
&= \|L\|_F \|R\|_F
\end{aligned} \tag{8}
$$

$\square$

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