Chapter 4.3

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坐标与投影

若$\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$为正交曲线坐标系，我们可以将任意向量$\vec{A}$表为$\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$之线性组合

$$\vec{A}=A_1\vec{e}_1+A_2\vec{e}_2+A_3\vec{e}_3 \qquad (4.3.1)$$

系数$\{A_1,A_2,A_3\}$就是向量$\vec{A}$在$\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$之分量或投影

$$A_1=\vec{A}\cdot\vec{e}_1,\quad A_2=\vec{A}\cdot\vec{e}_2,\quad A_3=\vec{A}\cdot\vec{e}_3 \qquad (4.3.2)$$

$\{A_1,A_2,A_3\}$可写得更清楚，假设

$$\vec{A}=(A_x,A_y,A_z)=A_x\vec{i}+A_y\vec{j}+A_z\vec{k} \qquad (4.3.3)$$

但是由(4.2.4)—(4.2.5)

$$\begin{aligned}
\vec{e}_1&=\frac{1}{h_1}\frac{\partial\vec{r}}{\partial u_1}=\frac{1}{h_1}\left(\frac{\partial x}{\partial u_1}\vec{i}+\frac{\partial y}{\partial u_1}\vec{j}+\frac{\partial z}{\partial u_1}\vec{k}\right)\\
\vec{e}_2&=\frac{1}{h_2}\frac{\partial\vec{r}}{\partial u_2}=\frac{1}{h_2}\left(\frac{\partial x}{\partial u_2}\vec{i}+\frac{\partial y}{\partial u_2}\vec{j}+\frac{\partial z}{\partial u_2}\vec{k}\right)\\
\vec{e}_3&=\frac{1}{h_3}\frac{\partial\vec{r}}{\partial u_3}=\frac{1}{h_3}\left(\frac{\partial x}{\partial u_3}\vec{i}+\frac{\partial y}{\partial u_3}\vec{j}+\frac{\partial z}{\partial u_3}\vec{k}\right)
\end{aligned} \qquad (4.3.4)$$

所以$\vec{A}$在$\vec{e}_1,\vec{e}_2,\vec{e}_3$之投影分别是

$$\begin{aligned}
\vec{A}\cdot\vec{e}_1&=A_1=\frac{1}{h_1}\vec{A}\cdot\frac{\partial\vec{r}}{\partial u_1}=\frac{1}{h_1}\left(A_x\frac{\partial x}{\partial u_1}+A_y\frac{\partial y}{\partial u_1}+A_z\frac{\partial z}{\partial u_1}\right)\\
\vec{A}\cdot\vec{e}_2&=A_2=\frac{1}{h_2}\vec{A}\cdot\frac{\partial\vec{r}}{\partial u_2}=\frac{1}{h_2}\left(A_x\frac{\partial x}{\partial u_2}+A_y\frac{\partial y}{\partial u_2}+A_z\frac{\partial z}{\partial u_2}\right)\\
\vec{A}\cdot\vec{e}_3&=A_3=\frac{1}{h_3}\vec{A}\cdot\frac{\partial\vec{r}}{\partial u_3}=\frac{1}{h_3}\left(A_x\frac{\partial x}{\partial u_3}+A_y\frac{\partial y}{\partial u_3}+A_z\frac{\partial z}{\partial u_3}\right)
\end{aligned} \qquad (4.3.5)$$

注解：

(1) 如果$\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$不是正交曲线坐标系，事情就没有这么容易，我们必须借助互反集合(4.2.8)，因为

$$\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}=\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}=\vec{r}_{u_1}\cdot(\vec{r}_{u_2}\times\vec{r}_{u_3})\neq 0$$

集合$\{\vec{r}_{u_1},\vec{r}_{u_2},\vec{r}_{u_3}\}$是线性独立 (linear independent)，我们可以将任意向量$\vec{a}$表为

$$\vec{a}=b_1\vec{r}_{u_1}+b_2\vec{r}_{u_2}+b_3\vec{r}_{u_3}$$

再与∇u_j, $j=1,2,3$, 做内积 (inner product)

$$\vec{a}\cdot\nabla u_j=b_1(\vec{r}_{u_1}\cdot\nabla u_j)+b_2(\vec{r}_{u_2}\cdot\nabla u_j)+b_3(\vec{r}_{u_3}\cdot\nabla u_j)=b_j$$

所以 $$\vec{a}=\sum_{j=1}^{3}(\vec{a}\cdot\nabla u_j)\vec{r}_{u_j} \quad \text{或} \quad \boxed{\vec{a}=\sum_{j=1}^{3}(\vec{a}\cdot\vec{r}_{u_j})\nabla u_j} \qquad 4.3.5'$$

对

$\vec{a} = \left(\sum_{j=1}^{3} (\vec{a}\cdot\nabla u_j)\vec{r}_{u_j}\right)$ ‖ $\vec{c} = \sum_{j=1}^{3} (\ \ )\nabla u_j$ ‖ 4.3.5

Left

由于 $\vec{a}\cdot\vec{b} = \left(\sum_{j=1}^{3} (\vec{b}\cdot\nabla u_j)\vec{r}_{u_j}\right)\cdot\vec{b}$

$= \sum_{j=1}^{3} (\vec{a}\cdot\nabla u_j)(\vec{r}_{u_j}\cdot\vec{b})$

又因 $\vec{c}\cdot\vec{b} = \left(\sum_{j=1}^{3} (\vec{a}\cdot\vec{r}_{u_j})\nabla u_j\right)\cdot\vec{b}$

$= \sum_{j=1}^{3} (\vec{a}\cdot\vec{r}_{u_j})(\nabla u_j\cdot\vec{b})$

$(\vec{a}-\vec{c})\cdot\vec{b} = 0$，$\vec{b}$ 不妨取为 $\vec{b}$，则 $\vec{a} = \vec{c}$

例题 4.3.1

试将向量 $\vec{A} = (z, -2x, y) = z\vec{i} - 2x\vec{j} + y\vec{k}$ 表示为圆柱坐标。

解：由例题 4.2.1 得
$\vec{i} = \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta$
$\vec{j} = \sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta$
$\vec{k} = \vec{e}_z$

所以 $\vec{A} = z\vec{i} - 2x\vec{j} + y\vec{k}$

$= z(\cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta) - 2r\cos\theta(\sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta) + r\sin\theta\,\vec{k}$

$= (z\cos\theta - 2r\cos\theta\sin\theta)\,\vec{e}_r - (z\sin\theta + 2r\cos^2\theta)\,\vec{e}_\theta + r\sin\theta\,\vec{e}_z$

在曲线坐标考虑质点的位置向量 $\vec{r} = \vec{r}(u_1, u_2, u_3)$，则由 (4.2.11) 得速度

$$\vec{v} = \frac{d\vec{r}}{dt} = h_1\dot{u}_1\vec{e}_1 + h_2\dot{u}_2\vec{e}_2 + h_3\dot{u}_3\vec{e}_3 \qquad 4.3.6$$

加速度可以假设为
$$\vec{a} = \frac{d\vec{v}}{dt} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$$

系数 $\{a_1, a_2, a_3\}$ 就是向量 $\vec{a}$ 在 $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ 之投影

$$a_1 = \vec{a}\cdot\vec{e}_1 = \frac{d\vec{v}}{dt}\cdot\frac{1}{h_1}\frac{\partial\vec{r}}{\partial u_1}$$
$$a_2 = \vec{a}\cdot\vec{e}_2 = \frac{d\vec{v}}{dt}\cdot\frac{1}{h_2}\frac{\partial\vec{r}}{\partial u_2}$$
$$a_3 = \vec{a}\cdot\vec{e}_3 = \frac{d\vec{v}}{dt}\cdot\frac{1}{h_3}\frac{\partial\vec{r}}{\partial u_3}$$

$$h_1 a_1 = \frac{d\vec{v}}{dt}\cdot\frac{\partial\vec{r}}{\partial u_1}$$
$$= \frac{d\left(\vec{v}\cdot\frac{\partial\vec{r}}{\partial u_1}\right)}{dt} - \vec{v}\cdot\frac{d\frac{\partial\vec{r}}{\partial u_1}}{dt}$$

$$= \frac{d\left(\vec{v}\cdot\frac{\partial \vec{r}}{\partial u_1}\right)}{dt} - \vec{v}\cdot\frac{d\,\frac{\partial \vec{r}}{\partial u_1}}{dt}$$

$$= \frac{d}{dt}\left(\vec{v}\cdot\frac{\partial \vec{r}}{\partial u_1}\right) - \vec{v}\cdot\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial u_1}\right) \qquad (4.3.7)$$

但是(4.3.6)对 $\dot{u}_1$ 微分,

$$\frac{\partial \vec{v}}{\partial \dot{u}_1} = h_1\vec{e}_1 = \frac{\partial \vec{r}}{\partial u_1} \qquad (4.3.8)$$

求 $\frac{\partial u_1}{\partial u_2} = 0$, $\Rightarrow$ $\frac{\partial \dot{u}_1}{\partial \dot{u}_2} = 0$

$\lim\limits_{t\to 0}\frac{f(t)}{g(t)} = \lim\limits_{t\to 0}\frac{f'(t)}{g'(t)}$ $\Rightarrow$ 求 $f'(t)\to 0$, $g'(t)\to 0$ 条件

又因

$$\frac{d}{dt}\frac{\partial \vec{r}}{\partial u_1} = \frac{\partial}{\partial u_1}\left(\frac{\partial \vec{r}}{\partial u_1}\right)\dot{u}_1 + \frac{\partial}{\partial u_2}\left(\frac{\partial \vec{r}}{\partial u_1}\right)\dot{u}_2 + \frac{\partial}{\partial u_3}\left(\frac{\partial \vec{r}}{\partial u_1}\right)\dot{u}_3 \qquad 4.3.9$$

$$= \frac{\partial}{\partial u_1}\left(\frac{\partial \vec{r}}{\partial u_1}\right)\dot{u}_1 + \frac{\partial}{\partial u_1}\left(\frac{\partial \vec{r}}{\partial u_2}\right)\dot{u}_2 + \frac{\partial}{\partial u_1}\left(\frac{\partial \vec{r}}{\partial u_3}\right)\dot{u}_3$$

$$= \frac{\partial}{\partial u_1}\frac{d\vec{r}}{dt} = \frac{\partial \vec{v}}{\partial u_1}$$

将由(4.3.5),(4.3.9),可以将(4.3.7)改写为

$$h_1 a_1 = \frac{d}{dt}\left(\vec{v}\cdot\frac{\partial \vec{v}}{\partial \dot{u}_1}\right) - \vec{v}\cdot\frac{\partial \vec{v}}{\partial u_1}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{u}_1}\left(\frac{|\vec{v}|^2}{2}\right) - \frac{\partial}{\partial u_1}\left(\frac{|\vec{v}|^2}{2}\right)$$

why?

我同理可得 a_2, a_3)

$$a_1 = \frac{1}{h_1}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{u}_1} - \frac{\partial T}{\partial u_1}\right)$$

$$a_2 = \frac{1}{h_2}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{u}_2} - \frac{\partial T}{\partial u_2}\right) \qquad (4.3.10)$$

$$a_3 = \frac{1}{h_2}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{u}_3} - \frac{\partial T}{\partial u_3}\right)$$

其中 T 是动能 (kinetic energy):

$$T = \frac{|\vec{v}|^2}{2} = \frac{1}{2}\left(h_1^2\dot{u}_1^2 + h_2^2\dot{u}_2^2 + h_3^2\dot{u}_3^2\right) \qquad 4.3.11$$

例题 4.3.2 (圆柱坐标)

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圆柱坐标下位置向量，速度和加速度表示为

$$\vec r=(x,y,z)=(r\cos\theta,\ r\sin\theta,\ z)=r\vec e_r+z\vec e_z$$

$$\dot{\vec r}=\frac{d\vec r}{dt}=v_r\vec e_r+v_\theta\vec e_\theta+v_z\vec e_z=\dot r\vec e_r+r\dot\theta\vec e_\theta+\dot z\vec e_z$$

$$\vec a=\frac{d\dot{\vec r}}{dt}=a_r\vec e_r+a_\theta\vec e_\theta+a_z\vec e_z=(\ddot r-r\dot\theta^2)\vec e_r+(r\ddot\theta+2\dot r\dot\theta)\vec e_\theta+\ddot z\vec e_z$$

解: 令 $\vec A=A_r\vec e_r+A_\theta\vec e_\theta+A_z\vec e_z$，则由例题 4.2.1 与 (4.3.5) 以及 (4.3.2) 得.

$$A_r=\vec A\cdot\vec e_r=(r\cos\theta,\ r\sin\theta,\ z)\cdot(\cos\theta,\ \sin\theta,\ 0)=r$$
$$A_\theta=\vec A\cdot\vec e_\theta=(r\cos\theta,\ r\sin\theta,\ z)\cdot(-\sin\theta,\ \cos\theta,\ 0)=0$$
$$A_z=\vec A\cdot\vec e_z=(r\cos\theta,\ r\sin\theta,\ z)\cdot(0,0,1)=z$$

故 $\vec r=r\vec e_r+z\vec e_z$ 另外因为 $\vec e_r=\cos\theta\,\vec i+\sin\theta\,\vec j$, $\vec e_\theta=-\sin\theta\,\vec i+\cos\theta\,\vec j$

所以

$$\frac{d}{dt}\vec e_r=(-\sin\theta\,\vec i+\cos\theta\,\vec j)\,\dot\theta=\dot\theta\,\vec e_\theta \qquad 4.3.12$$
$$\frac{d}{dt}\vec e_\theta=(-\cos\theta\,\vec i+\sin\theta\,\vec j)\,\dot\theta=-\dot\theta\,\vec e_r$$

故

$$\dot{\vec r}=\frac{d}{dt}\vec r=\frac{d}{dt}(r\vec e_r+z\vec e_z)$$
$$=\dot r\vec e_r+r\frac{d}{dt}\vec e_r+\dot z\vec e_z$$
$$=\dot r\vec e_r+r\dot\theta\vec e_\theta+\dot z\vec e_z$$

同理可证 $\vec a=(\ddot r-r\dot\theta^2)\vec e_r+(r\ddot\theta+2\dot r\dot\theta)\vec e_\theta+\ddot z\vec e_z$.

例题 4.3.3 (球坐标)

球坐标之 $\vec e_\rho$, $\vec e_\varphi$, $\vec e_\theta$ 满足微分方程

$$d\begin{pmatrix}\vec e_\rho\\ \vec e_\varphi\\ \vec e_\theta\end{pmatrix}=\begin{pmatrix}0 & d\varphi & \sin\varphi\, d\theta\\ -d\varphi & 0 & \cos\varphi\, d\theta\\ -\sin\varphi\, d\theta & -\cos\varphi\, d\theta & 0\end{pmatrix}\begin{pmatrix}\vec e_\rho\\ \vec e_\varphi\\ \vec e_\theta\end{pmatrix} \qquad (4.3.13)$$

解: 因为

$$\vec e_\rho=(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$
$$\vec e_\varphi=(\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi)$$
$$\vec e_\theta=(-\sin\theta,\ \cos\theta,\ 0)$$

对其微分

$$\vec{e}_\theta = (-\sin\theta, \cos\theta, 0)$$

直接微分

$$d\vec{e}_\rho = d\varphi(\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) + \sin\varphi\, d\theta(-\sin\theta, \cos\theta, 0)$$

$$= d\varphi\, \vec{e}_\varphi + \sin\varphi\, d\theta\, \vec{e}_\theta$$

同理可得其它各项.

例题 4.3.4

证明在球坐标下位置向量,速度与加速度表示为

$$\vec{r} = (x,y,z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) = \rho\,\vec{e}_\rho$$

$$\vec{v} = \dot\rho\,\vec{e}_\rho + \rho\dot\varphi\,\vec{e}_\varphi + \rho\dot\theta\sin\varphi\,\vec{e}_\theta$$

$$\vec{a} = (\ddot\rho - \rho\dot\varphi^2 - \rho\dot\theta^2\sin^2\varphi)\vec{e}_\rho + (\rho\ddot\varphi + 2\dot\rho\dot\varphi - \rho\dot\theta^2\sin\varphi\cos\varphi)\vec{e}_\varphi$$

$$+ (2\dot\rho\dot\theta\sin\varphi + 2\rho\dot\varphi\cos\varphi + \rho\ddot\theta\sin\varphi)\,\vec{e}_\theta$$

解: ...... □

由动力学自然引入了微分几何的活动标架,(moving frame)之概念, $\vec{e}_1, \vec{e}_2, \vec{e}_3$ 除了满足(4.2.20)之性质外,

$$d\vec{r} = h_1 du_1 \vec{e}_1 + h_2 du_2 \vec{e}_2 + h_3 du_3 \vec{e}_3 = \omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3$$

(上式 i.e. 4.2.20)

还有 $$d\vec{e}_i = \omega_{i1}\vec{e}_1 + \omega_{i2}\vec{e}_2 + \omega_{i3}\vec{e}_3, \quad i=1,2,3$$

或者表为矩阵之形式:

$$d\vec{r} = (\omega_1, \omega_2, \omega_3)\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}$$

$$d\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix} = \begin{pmatrix}\omega_{11} & \omega_{12} & \omega_{13}\\ \omega_{21} & \omega_{22} & \omega_{23}\\ \omega_{31} & \omega_{32} & \omega_{33}\end{pmatrix}\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}$$

$$= \Omega\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}, \quad \Omega = (\omega_{ij}) \qquad 4.3.14$$

因为 $\vec{e}_i\cdot\vec{e}_j = \delta_{ij}$, 所以 $d\vec{e}_i\cdot\vec{e}_j + \vec{e}_i\cdot d\vec{e}_j = 0$

因为 $\vec{e}_i\cdot\vec{e}_j=\delta_{ij}$，所以 $d\vec{e}_i\cdot\vec{e}_j+\vec{e}_i\cdot d\vec{e}_j=0$

$$(w_{i1}\vec{e}_1+w_{i2}\vec{e}_2+w_{i3}\vec{e}_3)\cdot\vec{e}_j+\vec{e}_i\cdot(w_{j1}\vec{e}_1+w_{j2}\vec{e}_2+w_{j3}\vec{e}_3)$$
$$=(w_{i1}+w_{j1})\vec{e}_1+(w_{i2}+w_{j2})\vec{e}_2+(w_{i3}+w_{j3})\vec{e}_3$$
$$=0$$

有 $w_{i1}+w_{j1}=0$，$w_{i2}+w_{j2}=0$，$w_{i3}+w_{j3}=0$

记 $w_{ij}+w_{ji}=0$ 则 $\Omega+\Omega^t=0$

容易验证 $w_{ii}=0$，我们还可以推得其它结构条件，因为

$$d(d\vec{r})=d(dx_1,dx_2,dx_3)=(0,0,0)$$

$\boxed{dx_1}$ $\boxed{常数}$?

$d(d\vec{r})=0$，得

$d(d\vec{r})=0$，得：

$$d(d\vec{r})=d(w_1,w_2,w_3)\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}-(w_1,w_2,w_3)\,d\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}$$

$$=\left[d(w_1,w_2,w_3)-(w_1,w_2,w_3)\Omega\right]\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}=\vec{0}$$

所以 $d(w_1,w_2,w_3)=(w_1,w_2,w_3)\Omega$ (4.3.15)

同理，因为 $d(d(\vec{e}_1,\vec{e}_2,\vec{e}_3)^t)=0$ 得：

$$0=d\Omega(\vec{e}_1,\vec{e}_2,\vec{e}_3)^t-\Omega\,d(\vec{e}_1,\vec{e}_2,\vec{e}_3)^t$$
$$=(d\Omega-\Omega^2)(\vec{e}_1,\vec{e}_2,\vec{e}_3)^t$$

所以 $d\Omega-\Omega^2=0$ $\boxed{(4.3.16)}$

对于 x 轴

$$d\,dx=d\,\boxed{1dx}=0\,dx$$

$\boxed{dx \text{ 对 } x \text{ 求导}}$

$$\frac{1\,d(x+\Delta x)-1\,dx}{\Delta x}$$

$$= \frac{0}{\Delta x} = 0$$

基础不牢，地动山摇

定理4.3.5

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$之微分满足之关系式：

$$\begin{cases} \dfrac{\partial \vec{e}_1}{\partial u_1} = -\dfrac{1}{h_2}\dfrac{\partial h_1}{\partial u_2}\vec{e}_2 - \dfrac{1}{h_3}\dfrac{\partial h_1}{\partial u_3}\vec{e}_3 \\ \dfrac{\partial \vec{e}_1}{\partial u_2} = \dfrac{1}{h_1}\dfrac{\partial h_2}{\partial u_1}\vec{e}_2 \\ \dfrac{\partial \vec{e}_1}{\partial u_3} = \dfrac{1}{h_1}\dfrac{\partial h_3}{\partial u_1}\vec{e}_3 \end{cases} \quad (4.3.17)$$

$$\begin{cases} \dfrac{\partial \vec{e}_2}{\partial u_1} = \dfrac{1}{h_2}\dfrac{\partial h_1}{\partial u_2}\vec{e}_1 \\ \dfrac{\partial \vec{e}_2}{\partial u_2} = -\dfrac{1}{h_3}\dfrac{\partial h_2}{\partial u_3}\vec{e}_3 - \dfrac{1}{h_1}\dfrac{\partial h_2}{\partial u_1}\vec{e}_1 \\ \dfrac{\partial \vec{e}_2}{\partial u_3} = \dfrac{1}{h_2}\dfrac{\partial h_3}{\partial u_2}\vec{e}_3 \end{cases} \quad (4.3.18)$$

$$\begin{cases} \dfrac{\partial \vec{e}_3}{\partial u_1} = \dfrac{1}{h_3}\dfrac{\partial h_1}{\partial u_3}\vec{e}_1 \\ \dfrac{\partial \vec{e}_3}{\partial u_2} = \dfrac{1}{h_3}\dfrac{\partial h_2}{\partial u_3}\vec{e}_2 \\ \dfrac{\partial \vec{e}_3}{\partial u_3} = -\dfrac{1}{h_1}\dfrac{\partial h_3}{\partial u_1}\vec{e}_1 - \dfrac{1}{h_2}\dfrac{\partial h_3}{\partial u_2}\vec{e}_2 \end{cases} \quad (4.3.19)$$

或者写为 $(\vec{i} \to \vec{j} \to \vec{k})$

$$\frac{\partial \vec{e}_i}{\partial u_i} = -\frac{1}{h_j}\frac{\partial h_i}{\partial u_j}\vec{e}_j - \frac{1}{h_k}\frac{\partial h_i}{\partial u_k}\vec{e}_k \quad (i,j,k)\text{是一置换} \quad (4.3.20)$$

$$\frac{\partial \vec{e}_i}{\partial u_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\vec{e}_j, \quad i \neq j$$

证明：因为 $\vec{e}_i \cdot \vec{e}_i = 1$，$i=1,2,3$，对 u_j 微分得

$$\frac{\partial \vec{e}_i}{\partial u_j} \cdot \vec{e}_i + \vec{e}_i \cdot \frac{\partial \vec{e}_i}{\partial u_j} = 0, \quad j=1,2,3$$

$$\frac{\partial \vec{e}_i}{\partial u_j}\cdot\vec{e}_i + \vec{e}_i\cdot\frac{\partial \vec{e}_i}{\partial u_j} = 0 , \quad j=1,2,3$$

i.e. $\frac{\partial \vec{e}_i}{\partial u_j}\cdot\vec{e}_i = 0$

所以 $\frac{\partial \vec{e}_i}{\partial u_j} \perp \vec{e}_i$，特别

$$\frac{\partial \vec{e}_1}{\partial u_2} \perp \vec{e}_1 , \quad \frac{\partial \vec{e}_2}{\partial u_1} \perp \vec{e}_2 \qquad (4.3.21)$$

其次是正交性

$$\frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial \vec{r}}{\partial u_2} = h_1\vec{e}_1\cdot h_2\vec{e}_2 = 0$$

两边对 u_3 微分

$$\frac{\partial}{\partial u_3}\left(\frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial \vec{r}}{\partial u_2}\right) = \frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial^2 \vec{r}}{\partial u_2\partial u_3} + \frac{\partial \vec{r}}{\partial u_2}\cdot\frac{\partial^2 \vec{r}}{\partial u_3\partial u_1} = 0 \qquad (4.3.22)$$

同理将 u_1, u_2, u_3 做排列 (permutation) 可得类似的式子

$$\frac{\partial}{\partial u_1}\left(\frac{\partial \vec{r}}{\partial u_2}\cdot\frac{\partial \vec{r}}{\partial u_3}\right) = \frac{\partial \vec{r}}{\partial u_2}\cdot\frac{\partial^2 \vec{r}}{\partial u_3\partial u_1} + \frac{\partial \vec{r}}{\partial u_3}\cdot\frac{\partial^2 \vec{r}}{\partial u_1\partial u_2} = 0$$

$$\frac{\partial}{\partial u_2}\left(\frac{\partial \vec{r}}{\partial u_3}\cdot\frac{\partial \vec{r}}{\partial u_1}\right) = \frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial^2 \vec{r}}{\partial u_2\partial u_3} + \frac{\partial \vec{r}}{\partial u_3}\cdot\frac{\partial^2 \vec{r}}{\partial u_2\partial u_1} = 0$$

三式相加得

$$2\left(\frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial^2 \vec{r}}{\partial u_2\partial u_3} + \frac{\partial \vec{r}}{\partial u_2}\cdot\frac{\partial^2 \vec{r}}{\partial u_3\partial u_1} + \frac{\partial \vec{r}}{\partial u_3}\cdot\frac{\partial^2 \vec{r}}{\partial u_1\partial u_2}\right) = 0$$

i.e. $$\frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial^2 \vec{r}}{\partial u_2\partial u_3} + \frac{\partial \vec{r}}{\partial u_2}\cdot\frac{\partial^2 \vec{r}}{\partial u_3\partial u_1} + \frac{\partial \vec{r}}{\partial u_3}\cdot\frac{\partial^2 \vec{r}}{\partial u_1\partial u_2} = 0 \qquad (4.3.23)$$

~~再~~减去等式 (4.3.22)，所以，

$$\frac{\partial \vec{r}}{\partial u_3}\cdot\frac{\partial^2 \vec{r}}{\partial u_1\partial u_2} = 0 \qquad (4.3.24)$$

但是

$$\frac{\partial^2 \vec{r}}{\partial u_1\partial u_2} = \frac{\partial}{\partial u_2}\left(\frac{\partial \vec{r}}{\partial u_1}\right) = \frac{\partial}{\partial u_2}(h_1\vec{e}_1)$$

$$= \frac{\partial}{\partial u_1}\left(\frac{\partial \vec{r}}{\partial u_2}\right) = \frac{\partial}{\partial u_1}(h_2\vec{e}_2) \qquad (4.3.25)$$

所以 (4.3.24) 告诉我们

$$\vec{e}_3 = \frac{\frac{\partial \vec{r}}{\partial u_3}}{\left|\frac{\partial \vec{r}}{\partial u_3}\right| = h_3} \Rightarrow \frac{\partial \vec{r}}{\partial u_3} = h_3\vec{e}_3$$

$$0 = \frac{\partial \vec{r}}{\partial u_3}\cdot\frac{\partial^2 \vec{r}}{\partial u_1\partial u_2} = h_3\vec{e}_3\cdot\frac{\partial^2 \vec{r}}{\partial u_1\partial u_2}$$

$$= h_3\,\vec{e}_3\cdot\frac{\partial(h_1\vec{e}_1)}{\partial u_1}$$

h_1, h_2, h_3 $\boxed{\text{function}}$

$$= h_3 \, \vec{e}_3 \cdot \frac{\partial (h_1 \vec{e}_1)}{\partial x_1}$$

h_1, h_2, h_3 $\boxed{\text{function}}$

$$= h_3 \, \vec{e}_3 \cdot \frac{\partial (h_2 \vec{e}_2)}{\partial x_1}$$

$$\frac{\partial h_1 \vec{e}_1}{\partial x_2} \perp \vec{e}_3, \quad \frac{\partial h_2 \vec{e}_2}{\partial x_1} \perp \vec{e}_3$$

$h_1 \neq 0$

$$\left(h_1 \boxed{\frac{\partial \vec{e}_1}{\partial x_2}} + \vec{e}_1 \frac{\partial h_1}{\partial x_2}\right) \cdot \vec{e}_3 = 0 \iff h_1 \frac{\partial \vec{e}_1}{\partial x_2} \cdot \vec{e}_3 + \underbrace{\vec{e}_1 \cdot \vec{e}_3}_{0} \frac{\partial h}{\partial x_2} = 0$$

$$\Rightarrow \frac{\partial \vec{e}_1}{\partial x_2} \cdot \vec{e}_3 = 0$$

$$\left(h_2 \frac{\partial \vec{e}_2}{\partial x_1} + \vec{e}_2 \frac{\partial h_2}{\partial x_1}\right) \cdot \vec{e}_3 = 0$$

$$\Rightarrow \frac{\partial \vec{e}_2}{\partial x_1} \cdot \vec{e}_3 = 0$$

故 $\frac{\partial \vec{e}_1}{\partial x_2} \perp \vec{e}_3, \quad \frac{\partial \vec{e}_2}{\partial x_1} \perp \vec{e}_3$ (4.3.26)

由 (4.3.21)，(4.3.26) 得

$$\frac{\partial \vec{e}_1}{\partial x_2} \perp \vec{e}_1, \quad \frac{\partial \vec{e}_2}{\partial x_1} \perp \vec{e}_2 \qquad \frac{\partial \vec{e}_1}{\partial x_2} \parallel \vec{e}_2, \quad \frac{\partial \vec{e}_2}{\partial x_1} \parallel \vec{e}_1$$

(4.3.25) 可以展开为

$$\frac{\partial (h_1 \vec{e}_1)}{\partial x_2} = \frac{\partial (h_2 \vec{e}_2)}{\partial x_1}$$

$$\Downarrow$$

$$h_1 \underbrace{\frac{\partial \vec{e}_1}{\partial x_2}}_{\gamma \vec{e}_2} + \vec{e}_1 \frac{\partial h_1}{\partial x_2} = h_2 \underbrace{\frac{\partial \vec{e}_2}{\partial x_1}}_{n \vec{e}_1} + \vec{e}_2 \frac{\partial h_2}{\partial x_1}$$

由于 $\vec{e}_1, \vec{e}_2$ 线性无关 则不存在 γ, s.t. $\vec{e}_1 = \gamma \vec{e}_2$

但 $\vec{e}_1, \vec{e}_2$ 是线性独立的 故

$$\frac{\partial \vec{e}_1}{\partial x_2} = \frac{1}{h_1} \frac{\partial h_2}{\partial x_1} \vec{e}_2, \quad \frac{\partial \vec{e}_2}{\partial x_1} = \frac{1}{h_2} \frac{\partial h_1}{\partial x_2} \vec{e}_1$$

(下标调整，基调整)

这就是 (4.3.17) 的第二、三式，同理可得 (4.3.18)，(4.3.19) 的第二、三式，另外，

$$\frac{\partial \vec{e}_1}{\partial x_1} = \frac{\partial (\vec{e}_2 \times \vec{e}_3)}{\partial x_1} = \frac{\partial \vec{e}_2}{\partial x_1} \times \vec{e}_3 + \vec{e}_2 \times \frac{\partial \vec{e}_3}{\partial x_1}$$

$$\frac{\partial \vec{e}_1}{\partial u_1} = \frac{\partial(\vec{e}_2\times\vec{e}_3)}{\partial u_1} = \frac{\partial \vec{e}_2}{\partial u_1}\times\vec{e}_3 + \vec{e}_2\times\frac{\partial \vec{e}_3}{\partial u_1}$$
$$= \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\vec{e}_1\times\vec{e}_3 + \vec{e}_2\times\frac{1}{h_3}\frac{\partial h_1}{\partial u_3}\vec{e}_1$$
$$= -\frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\vec{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial u_3}\vec{e}_3$$

+ 1 2 3
− 2 1 3

这是(4.3.17)的第一式，同理可得(4.3.18)，(4.3.19)的第一式。

例题 4.3.6（圆柱座标）

圆柱座标系之单位向量 $\vec{e}_r$，$\vec{e}_\theta$，$\vec{e}_z$ 的各阶偏导数为

$$\boxed{\begin{aligned}\vec{e}_r &= \cos\theta\,\vec{i} + \sin\theta\,\vec{j}\\ \vec{e}_\theta &= -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\\ \vec{e}_z &= \vec{k}\end{aligned}}$$

$$\frac{\partial \vec{e}_r}{\partial r} = \vec{0},\quad \frac{\partial \vec{e}_r}{\partial \theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} = \vec{e}_\theta,\quad \frac{\partial \vec{e}_r}{\partial z} = \vec{0} \qquad (4.3.28a)$$

$$\frac{\partial \vec{e}_\theta}{\partial r} = \vec{0},\quad \frac{\partial \vec{e}_\theta}{\partial \theta} = -\cos\theta\,\vec{i} - \sin\theta\,\vec{j} = -\vec{e}_r,\quad \frac{\partial \vec{e}_\theta}{\partial z} = \vec{0} \qquad (4.3.28b)$$

$$\frac{\partial \vec{e}_z}{\partial r} = \vec{0},\quad \frac{\partial \vec{e}_z}{\partial \theta} = \vec{0},\quad \frac{\partial \vec{e}_z}{\partial z} = \vec{0} \qquad (4.3.28c)$$

最著名的活动标架是 Frenet 标架

定性 4.3.7（Frenet 标架）已知 $\vec{t}$ 是曲线之单位切向量，$\vec{n}$ 是单位法向量，次法向量 $\vec{b} = \vec{t}\times\vec{n}$ 满足关系式：

$$\frac{d}{ds}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix} = \begin{pmatrix}0 & k & 0\\ -k & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix} \qquad (4.3.29)$$

$$\text{as} \quad (\vec{\ }) \quad (0 \quad -\tau \quad 0)(\vec{\ })$$

其中 κ 之曲率 (curvature)，τ 之扭率 (torsion).

定理 4.3.8 (梯度) 已知 $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ 为正交，f 为一函数，则其梯度 (gradient)

$$
\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \\
&= \frac{\partial f}{\partial x_1}\vec{i} + \frac{\partial f}{\partial x_2}\vec{j} + \frac{\partial f}{\partial x_3}\vec{k} \qquad (4\text{-}3\text{-}30) \\
&= \frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\vec{e}_3
\end{aligned}
$$

证明：因为 $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ 是一正交基底，因此可以考虑 ∇f 在这三个方向之投影

$$
\begin{aligned}
\nabla f \cdot \vec{e}_1 &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \cdot \left(\frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1}\right) \\
&= \frac{1}{h_1}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \cdot \left(\frac{\partial x_1}{\partial u_1}, \frac{\partial x_2}{\partial u_1}, \frac{\partial x_3}{\partial u_1}\right) \\
&= \frac{1}{h_1}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_1} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial u_1}\right) \\
&= \underline{\frac{1}{h_1}\frac{\partial f}{\partial u_1}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla f \cdot \vec{e}_2 &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \cdot \left(\frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2}\right) \\
&= \frac{1}{h_2}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_2} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_2} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial u_2}\right) \\
&= \frac{1}{h_2}\frac{\partial f}{\partial u_2}
\end{aligned}
$$

$$
\begin{aligned}
\nabla f \cdot \vec{e}_3 &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \cdot \left(\frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3}\right) \\
&= \frac{1}{h_3}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_3} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_3} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial u_3}\right) \\
&= \frac{1}{h_3}\frac{\partial f}{\partial u_3}
\end{aligned}
$$

故得证.

由梯度 (gradient) 直接连想就是方向导数 (directional derivative):

$$\frac{df}{ds} = \nabla f \cdot \vec{t} \qquad \left[\ \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}\ \right]$$

$$\frac{df}{ds} = \nabla f\cdot\vec{t}$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$\vec{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$\vec{t}$ 是弧长方向之单位切向量

因此，若能计算方向导数 $\frac{df}{ds}$ 就可推得梯度 ∇f

$$\frac{df}{ds} = \frac{\partial f}{\partial u_1}\frac{du_1}{ds} + \frac{\partial f}{\partial u_2}\frac{du_2}{ds} + \frac{\partial f}{\partial u_3}\frac{du_3}{ds}$$

另，单位切向量

$$\vec{t} = \frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial u_1}\frac{du_1}{ds} + \frac{\partial \vec{r}}{\partial u_2}\frac{du_2}{ds} + \frac{\partial \vec{r}}{\partial u_3}\frac{du_3}{ds} = h_1\vec{e}_1\frac{du_1}{ds} + h_2\vec{e}_2\frac{du_2}{ds} + h_3\vec{e}_3\frac{du_3}{ds} \quad (4.3.3)$$

利用这关系则会将方向导数 $\frac{df}{ds}$ 表示为

$$\frac{df}{ds} = \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\vec{e}_3\right)\cdot\left(h_1\vec{e}_1\frac{du_1}{ds} + h_2\vec{e}_2\frac{du_2}{ds} + h_3\vec{e}_3\frac{du_3}{ds}\right)$$

$$= \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\vec{e}_3\right)\cdot\vec{t}$$

右边第一式就是 ∇f.

注解：

1. 由量纲分析的角度而言，在正交曲线坐标系统 h_i 必须配合 du_i，$[h_i du_i] = [ds_i] = L$，为着保持量纲平恒，梯度必然有 $\frac{1}{h_i}\frac{\partial}{\partial u_i}$，$i=1,2,3$，而且可以验证各项之量纲(因次)与 $[\nabla f] = \frac{[f]}{L}$ 相同

$$\left[\frac{1}{h_i}\frac{\partial f}{\partial u_i}\vec{e}_i\right] = \frac{1}{[h_i]}\left[\frac{\partial f}{\partial u_i}\right]L^0 = \frac{[f]}{L}, \quad i=1,2,3$$

2. 根据梯度的性质，∇f 在曲线坐标上之投影分别是该方向的方向导数

2. 根据梯度的性质，∇f 在曲线坐标上之投影分别是该方向的方向导数

$\frac{df}{ds_1}$，$\frac{df}{ds_2}$，$\frac{df}{ds_3}$，因此由定理 4.3.8 可得

$$\nabla f\cdot\vec{e}_1=\frac{df}{ds_1}=\frac{1}{h_1}\frac{\partial f}{\partial u_1}$$

$$\nabla f\cdot\vec{e}_2=\frac{df}{ds_2}=\frac{1}{h_2}\frac{\partial f}{\partial u_2}$$

$$\nabla f\cdot\vec{e}_3=\frac{df}{ds_3}=\frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

s_1, s_2, s_3 ?

$\boxed{4.2.2}$

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$ds_i=h_i\,du_i$

换句话说

$$\nabla f=\frac{df}{ds_1}\vec{e}_1+\frac{df}{ds_2}\vec{e}_2+\frac{df}{ds_3}\vec{e}_3\qquad(4.3.32)$$

3. 由梯度之公式 (4.3.30)，分别令 $f=u_1, u_2, u_3$ 得：

$$\boxed{\nabla f=\frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{e}_1+\frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{e}_2+\frac{1}{h_3}\frac{\partial f}{\partial u_3}\vec{e}_3}$$

$$\nabla u_1=\frac{1}{h_1}\vec{e}_1,\quad \nabla u_2=\frac{1}{h_2}\vec{e}_2,\quad \nabla u_3=\frac{1}{h_3}\vec{e}_3\qquad(4.3.33)$$

坐标之梯度正是曲空间之基底，而且 $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ 满足右手法则，所以

$$\vec{e}_1=\vec{e}_2\times\vec{e}_3=h_2h_3(\nabla u_2\times\nabla u_3)$$

$$\vec{e}_2=\vec{e}_3\times\vec{e}_1=h_3h_1(\nabla u_3\times\nabla u_1)\qquad(4.3.34)$$

$$\vec{e}_3=\vec{e}_1\times\vec{e}_2=h_1h_2(\nabla u_1\times\nabla u_2)$$

(4) 如果 $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ 不是正交曲线坐标系，

$$\nabla f=\sum_{j=1}^{3}\underline{(\nabla f)\cdot\vec{r}_{u_j}}\,\nabla_{u_j}$$

∇u_j 线性无关

$$\nabla f=a_1\boxed{\nabla u_1}+a_2\nabla u_2+a_3\nabla u_3$$

$$\nabla f\cdot\frac{\partial\vec{r}}{}\qquad \nabla u_1\cdot\frac{\partial\vec{r}}{}+\nabla u_2\cdot\frac{\partial\vec{r}}{\partial u}+\nabla u_3\cdot\frac{\partial\vec{r}}{\partial u}$$

$$\nabla f\cdot\frac{\partial\vec r}{\partial u_1} \quad \underline{\nabla u_1\cdot\frac{\partial\vec r}{\partial u_1}} + \underline{\nabla u_2\cdot\frac{\partial\vec r}{\partial u_1}} + \nabla u_3\cdot\frac{\partial\vec r}{\partial u_1}$$

$$\underline{\frac{\partial u_1}{\partial x_1}\frac{\partial x_1}{\partial u_1}+\frac{\partial u_1}{\partial x_2}\frac{\partial x_2}{\partial u_1}+\frac{\partial u_1}{\partial x_3}\frac{\partial x_3}{\partial u_1}} \qquad \frac{\partial u_2}{\partial u_1}=0 \qquad \frac{\partial u_3}{\partial u_1}=0$$

$$\frac{\partial u_1}{\partial u_1}=1$$

故

$$\nabla f=\sum_{j=1}^{3}(\nabla f)\cdot\vec r_{u_j}\ \nabla u_j$$

$$=\sum_{j=1}^{3}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_j}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_j}+\frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial u_j}\right)\nabla u_j$$

$$=\sum_{j=1}^{3}\frac{\partial f}{\partial u_j}\nabla u_j$$

因此计算梯度 ∇f 只需计算各个坐标曲线坐标之梯度 ∇u_j, $j=1,2,3$ - - - -

而 ∇u_j 可由 (4.2.9) 计算而得.

(5) 由 (4.3.30) 可以引进 Hamilton 算子

$$\nabla=\vec i\frac{\partial}{\partial x}+\vec j\frac{\partial}{\partial y}+\vec k\frac{\partial}{\partial z}$$

$$=\vec i\frac{\partial}{\partial x_1}+\vec j\frac{\partial}{\partial x_2}+\vec k\frac{\partial}{\partial x_3} \qquad (4.3.35)$$

$$=\frac{\vec e_1}{h_1}\frac{\partial}{\partial u_1}+\frac{\vec e_2}{h_2}\frac{\partial}{\partial u_2}+\frac{\vec e_3}{h_3}\frac{\partial}{\partial u_3}$$

这是一个具有向量与微分双重性质的算符, 它就是向量场 (vector field)

~~若 $f(u,v,w)=uv+w^2$ 则由 (4.3.35)~~

$$\cancel{\nabla f=\frac{\vec e_u}{h_1}\boxed{v}+\frac{\vec e_v}{h_2}\boxed{u}+\frac{\vec e_w}{h_3}\boxed{2w}}$$

定理 4.3.9 (散度) 已知向量 $\vec A=A_x\vec i+A_y\vec j+A_z\vec k$

$$=A_1\vec e_1+A_2\vec e_2+A_3\vec e_3,$$

则其散度

$$\mathrm{div}\,\vec A=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

$$=\frac{1}{h_1h_2h_3}\left[\left(\frac{\partial}{\partial u_1}h_2h_3A_1\right)+\left(\frac{\partial}{\partial u_2}h_3h_1A_2\right)+\frac{\partial}{\partial u_3}\left(h_1h_2A_3\right)\right] \qquad (4.3.36)$$

$$= \frac{1}{h_1 h_2 h_3}\left[\left(\frac{\partial}{\partial u_1} h_2 h_3 A_1\right) + \left(\frac{\partial}{\partial u_2} h_3 h_1 A_2\right) + \frac{\partial}{\partial u_3}\left(h_1 h_2 A_3\right)\right]$$

证明：因为散度是一线性算子（linear operator）

$$\operatorname{div}\vec{A} = \operatorname{div}\left(A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3\right)$$
$$= \operatorname{div}(A_1\vec{e}_1) + \operatorname{div}(A_2\vec{e}_2) + \operatorname{div}(A_3\vec{e}_3)$$

所以我们可以先计算三个分量，首先由梯度之公式

$$\nabla u_1 = \frac{1}{h_1}\vec{e}_1,\quad \nabla u_2 = \frac{1}{h_2}\vec{e}_2,\quad \nabla u_3 = \frac{1}{h_3}\vec{e}_3$$

$$\vec{e}_1 = \vec{e}_2 \times \vec{e}_3 = h_2 h_3\left(\nabla u_2 \times \nabla u_3\right)$$

第1个分量为

$$\operatorname{div}(A_1\vec{e}_1) = \operatorname{div}\left[\underbrace{A_1 h_2 h_3}_{f}\ \underbrace{\left(\nabla u_2 \times \nabla u_3\right)}_{\vec{A}}\right]$$

$$= \underline{A_1 h_2 h_3 \operatorname{div}\left(\nabla u_2 \times \nabla u_3\right)} + \nabla\left(A_1 h_2 h_3\right)\cdot\left(\nabla u_2 \times \nabla u_3\right)$$

其中 $\operatorname{div}\left(\nabla u_2 \times \nabla u_3\right)$

$$= \nabla u_3 \cdot \underbrace{\operatorname{curl}\nabla u_2}_{0} - \nabla u_2 \cdot \underbrace{\operatorname{curl}\nabla u_3}_{0} \qquad \nabla\times(\nabla u_2) = \vec{0}$$

$$= 0 - 0$$
$$= 0$$

$$\nabla\left(A_1 h_2 h_3\right)\cdot\left(\nabla u_2 \times \nabla u_3\right)$$

$$\vec{e}_2 = h_2 \nabla u_2$$
$$\vec{e}_3 = h_3 \nabla u_3$$

$$= \frac{1}{h_2 h_3}\left(\vec{e}_2 \times \vec{e}_3\right)\cdot \nabla A_1 h_2 h_3$$

$$\nabla f = \frac{\partial f}{\partial x_1}\vec{i} + \frac{\partial f}{\partial x_2}\vec{j} + \frac{\partial f}{\partial x_3}\vec{k}$$
$$= \frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{e}_1 + \frac{\vec{e}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\vec{e}_3}{h_3}\frac{\partial f}{\partial u_3}$$

$$= \frac{1}{h_2 h_3}\underbrace{\vec{e}_2\times\vec{e}_3}_{\vec{e}_1}\cdot\left(\frac{1}{h_1}\frac{\partial A_1 h_2 h_3}{\partial u_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial A_1 h_2 h_3}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial A_1 h_2 h_3}{\partial u_3}\vec{e}_3\right)$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial A_1 h_2 h_3}{\partial u_1}$$

故 $\operatorname{div} A_1\vec{e}_1 = \frac{1}{h_1 h_2 h_3}\frac{\partial A_1 h_2 h_3}{\partial u_1} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}\left(A_1 h_2 h_3\right)$ ... A

$\frac{1}{h_1h_2h_3}\ \frac{\partial}{\partial u_1}$

故 $\operatorname{div} A_1\vec{e}_1 = \frac{1}{h_1h_2h_3}\frac{\partial A_1h_2h_3}{\partial u_1} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(A_1h_2h_3)$

	A
23	1
31	2
12	3

同理可得: $\operatorname{div}(A_2\vec{e}_2) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}(h_3h_1A_2)$

$$\operatorname{div}(A_3\vec{e}_3) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}(h_1h_2A_3)$$

整体之和就是散度 $\operatorname{div}\vec{A}$ 之公式.

注释:

(1) 因为 h_i 同 du_i 组合, 为着保持量纲平衡中的散度, 例如第一项

$$\frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(h_2h_3A_1)$$

分子分母的 h_2h_3 相互抵消, 而 h_1 则正好配合 $\frac{\partial}{\partial u_1}$, 而且可以保证其量纲(同次)

$$\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(h_2h_3A_1)\right] = [A_1]/L$$

与 $[\operatorname{div}\vec{A}] = [A]/L$ 相同.

(2) 在证明过程中我们用了向量公式 (thm 2.7.9 p 135)

$$\operatorname{div}(f\vec{A}) = f\operatorname{div}\vec{A} + \vec{A}\cdot\nabla f$$
$$\operatorname{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\operatorname{curl}\vec{A} - \vec{A}\cdot\operatorname{curl}\vec{B}$$
$$\operatorname{curl}\nabla f = \nabla\times(\nabla f) = 0$$

梯度公式 4.3.36

$$\boxed{\begin{aligned}\nabla f &= \frac{\partial f}{\partial x_1}\vec{i} + \frac{\partial f}{\partial x_2}\vec{j} + \frac{\partial f}{\partial x_3}\vec{k}\\ &= \frac{\vec{e}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\vec{e}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\vec{e}_3}{h_3}\frac{\partial f}{\partial u_3}\end{aligned}}$$

对于不是正交曲线坐标系也成立, 因为证明过程只用到了定义(偏导)集合的性质.

(3) 我们也可以运用 Hamilton 算子来计算

$$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{A}$$

$$= \vec{i}\cdot\frac{\partial\vec{A}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{A}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{A}}{\partial z}$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

同理借由定理4.3.9 可得

$$\frac{\vec{e}_1}{h_1}\cdot\frac{\partial\vec{A}}{\partial u_1} + \frac{\vec{e}_2}{h_2}\cdot\frac{\partial\vec{A}}{\partial u_2} + \frac{\vec{e}_3}{h_3}\cdot\frac{\partial\vec{A}}{\partial u_3}$$

$$= \nabla u_1\cdot\frac{\partial\vec{A}}{\partial u_1} + \nabla u_2\cdot\frac{\partial\vec{A}}{\partial u_2} + \nabla u_3\cdot\frac{\partial\vec{A}}{\partial u_3}$$

$$= \frac{\partial u_1}{\partial x}\frac{\partial A_1}{\partial u_1} + \frac{\partial u_1}{\partial y}\frac{\partial A_2}{\partial u_1} + \frac{\partial u_1}{\partial z}\frac{\partial A_3}{\partial u_1}$$

$$+ \frac{\partial u_2}{\partial x}\frac{\partial A_1}{\partial u_2} + \frac{\partial u_2}{\partial y}\frac{\partial A_2}{\partial u_2} + \frac{\partial u_2}{\partial z}\frac{\partial A_3}{\partial u_2}$$

$$+ \frac{\partial u_3}{\partial x}\frac{\partial A_1}{\partial u_3} + \frac{\partial u_3}{\partial y}\frac{\partial A_2}{\partial u_3} + \frac{\partial u_3}{\partial z}\frac{\partial A_3}{\partial u_3}$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

故 $$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \frac{\vec{e}_1}{h_1}\cdot\frac{\partial\vec{A}}{\partial u_1} + \frac{\vec{e}_2}{h_2}\cdot\frac{\partial\vec{A}}{\partial u_2} + \frac{\vec{e}_3}{h_3}\cdot\frac{\partial\vec{A}}{\partial u_3}$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_2h_1A_3)\right]$$

定理4.3.10 (旋度)

已知向量 $\vec{A} = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$ 则其旋度等于

$$\operatorname{curl}\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \qquad (4.3.37)$$

证明: 因为旋度是线性算子, 而且 (note thm 2.7.9)

$$\operatorname{curl}(f\vec{A}) = f\cdot\operatorname{curl}\vec{A} + \nabla f\times\vec{A}, \quad \operatorname{curl}(\nabla f) = \vec{0}$$

$$\operatorname{curl}(A_1\vec{e}_1) = \operatorname{curl}\left(\underbrace{A_1h_1}_{f}\ \underbrace{\nabla u_1}_{\vec{A}}\right)$$

$$\text{curl}(A_1\vec e_1) = \text{curl}\,(\underbrace{A_1 h_1}_{f}\ \underbrace{\nabla u_1}_{\vec g})$$

$$= \underbrace{A_1h_1\,\text{curl}(\nabla u_1)}_{0} + (\nabla A_1h_1)\times\nabla u_1$$

$$= (\nabla A_1h_1)\times\nabla u_1$$

$$= \left(\frac{\vec e_1}{h_1}\frac{\partial A_1h_1}{\partial u_1} + \underline{\frac{1}{h_2}\frac{\partial A_1h_1}{\partial u_2}\vec e_2} + \frac{1}{h_3}\frac{\partial A_1h_1}{\partial u_3}\vec e_3\right)\times \underline{\frac{\vec e_1}{h_1}}$$

$$= -\frac{\vec e_1}{h_1}\times\frac{1}{h_2}\frac{\partial A_1h_1}{\partial u_2}\vec e_2 + \frac{1}{h_3}\frac{\partial A_1h_1}{\partial u_3}\vec e_3\times\frac{\vec e_1}{h_1}$$

$$= \frac{1}{h_3h_1}\frac{\partial A_1h_1}{\partial u_3}\vec e_2 - \frac{1}{h_1h_2}\frac{\partial A_1h_1}{\partial u_2}\vec e_3$$

$$= \frac{1}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1)\vec e_2 - \frac{1}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)\vec e_3$$

同理可得

$$\text{curl}(A_2\vec e_2) = \frac{1}{h_1h_2}\frac{\partial}{\partial u_1}(A_2h_2)\vec e_3 - \frac{1}{h_2h_3}\frac{\partial}{\partial u_3}(A_2h_2)\vec e_1$$

$$\text{curl}(A_3\vec e_3) = \frac{1}{h_2h_3}\frac{\partial}{\partial u_2}(A_3h_3)\vec e_1 - \frac{1}{h_3h_1}\frac{\partial}{\partial u_1}(A_3h_3)\vec e_2$$

三项相加，故得证

注解：

1. 因为 h_i 必须配合 du_i，为着保持量纲平衡度的放度，例如第一项（$\vec e_1$ 方向）

$$\frac{1}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1)\vec e_2 - \frac{1}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)\vec e_3$$

分子分母的 h_3, h_2 分别被也抵消而 $\frac{1}{h_1}$ 则正好配合 $\frac{\partial}{\partial u_1}$，而且可以验证其量纲（因次）等于 $\frac{[A_1]}{L}$ 与 $[\text{curl}\,\vec A] = \frac{[A]}{L}$ 相同，至于正负号则由是否满足右手法则来决定

$$\frac{1}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1)\vec e_2 \qquad 1\to2\to3 \qquad +$$

$$\frac{1}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)\vec e_3 \qquad 1\to3\to2 \qquad -$$

(2) 同样的理由旋度公式 (4.3.37) 对于不是正交曲线坐标系也成立，因为证明过程是用到正交集合的性质，此时 (4.3.37) 可以改写为

$$\operatorname{curl}\vec{A} = \frac{1}{J}\begin{vmatrix} \vec{r}_{u_1} & \vec{r}_{u_2} & \vec{r}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ \vec{A}\cdot\vec{r}_{u_1} & \vec{A}\cdot\vec{r}_{u_2} & \vec{A}\cdot\vec{r}_{u_3} \end{vmatrix} \quad (4.3.37')$$

4.3.10 (Laplace 算子) Laplace 算子可以表示为

$$\Delta f = \operatorname{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right] \quad 4.3.38$$

证明：由 (4.3.30) 知 梯度

$$\nabla f = \underline{\frac{1}{h_1}\frac{\partial f}{\partial u_1}}\vec{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\vec{e}_3$$

再由散度 (4.3.36)

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

得：

$$\Delta f = \operatorname{div}(\underline{\nabla f}) = \nabla\cdot(\nabla f)$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}h_2h_3, \frac{\partial}{\partial u_2}h_3h_1, \frac{\partial}{\partial u_3}h_1h_2\right]\cdot$$

$$\left[\frac{\vec{e}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\vec{e}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\vec{e}_3}{h_3}\frac{\partial f}{\partial u_3}\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(h_2h_3\frac{1}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(h_3h_1\frac{1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(h_1h_2\frac{1}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1} + \frac{\partial}{\partial u_2}\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2} + \frac{\partial}{\partial u_3}\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right]$$

梯度算子: $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$$= \frac{\vec{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\vec{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\vec{e}_3}{h_3}\frac{\partial}{\partial u_3}$$

散度算子: $\operatorname{div}\vec{A} = \nabla\cdot$

$A = (A_x, A_y, A_z)$
$= (A_1, A_2, A_3)$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(A_x, A_y, A_z)$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\underset{A_1}{(h_2h_3)} + \frac{\partial}{\partial u_2}\underset{A_2}{h_3h_1} + \frac{\partial}{\partial u_3}\underset{A_3}{h_2h_1}\right].$$

旋度算子: $\operatorname{curl}\vec{A} = \nabla\times\vec{A}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \underline{A_x} & \underline{A_y} & \underline{A_z} \end{vmatrix}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \frac{1}{h_1h_2h_3}\cdot\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1\underline{A_1} & h_2\underline{A_2} & h_3\underline{A_3} \end{vmatrix}$$

$$\frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_1 & h_2 \vec{e}_2 & h_3 \vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 \div & h_2 \div & h_3 \div \end{vmatrix}$$