Green's identities

The Green identities for the Laplacian lead directly to the maximum principle and to Dirichlet's principle about minimizing the energy. The Green's function is a kind of universal solution for harmonic functions in a domain. All other harmonic functions can be expressed in terms of it. Combined with the method of reflection, the Green's function leads in a very direct way to the solution of boundary problems in special geometries. George Green was interested in the new phenomena of electricity and magnetism in the early 19th century.

格林第一公式.

$u(x,y,z) \in \mathbb{R}$
$v(x,y,z) \in \mathbb{R}$

$$(v u_x)_x = v_x u_x + v u_{xx} \quad ①$$

$$(v u_y)_y = v_y u_y + v u_{yy} \quad ②$$

$$(v u_z)_z = v_z u_z + v u_{zz} \quad ③$$

left hand

$$\boxed{\begin{aligned} v\nabla u &= v\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \\ \nabla \cdot (v\nabla u) &= \frac{\partial (v u_x)}{\partial x} + \frac{\partial (v u_y)}{\partial y} + \frac{\partial (v u_z)}{\partial z} \end{aligned}}$$

right hand.

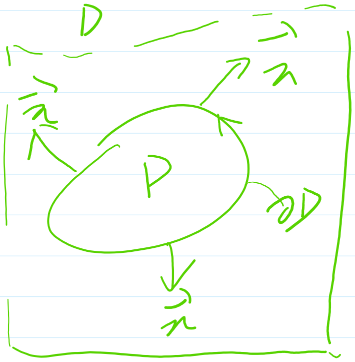
$$v_x u_x + v_y u_y + v_z u_z = \nabla v \cdot \nabla u$$

$$v_x u_x + v_y u_y + v_z u_z = \nabla v \cdot \nabla u$$
$$v u_{xx} + v u_{yy} + v u_{zz} = v \Delta u$$

所以. $\underbrace{\nabla \cdot (v \nabla u)}_{div} = \nabla v \cdot \nabla u + v \Delta u$

通过高斯散度定理

$$\iiint_D div(v \nabla u) dV = \iint_{\partial D} (v \nabla u) \cdot d\vec{S}$$



$$= \iint_{\partial D} v \nabla u \cdot \vec{n} \, dS$$
$$= \iint_{\partial D} v (\nabla u \cdot \vec{n}) \, dS$$
$$= \iint_{\partial D} v \frac{\partial u}{\partial \vec{n}} \, dS$$

所以我们有:

$$\iint_{\partial D} v \frac{\partial u}{\partial \vec{n}} dS = \iiint_D \nabla v \cdot \nabla u \, dV + \iiint_D v \Delta u \, dV \quad \boxed{G1}$$

格林第一公式的应用:

$v = 1$ 时

$$\iint_{\partial D} \frac{\partial u}{\partial \vec{n}} dS = \iiint_D \Delta u \, dV \quad **$$

Neumann 问题:

$$\begin{cases} \Delta u = f(x) & \text{in } D \\ \frac{\partial u}{\partial \vec{n}} = h(x) & \text{on } \partial D \end{cases}$$

通过 ** $\iint_{\partial D} \frac{\partial u}{\partial n} dS = \iiint_D \Delta u \, dV$

有: $\iint_{\partial D} h(x) dS = \iiint_D f(x) dV$

上式说明 Neumann 问题若有解, 则 $h(x)$, $f(x)$ 需

添上上述要求.

Neumann问题的解的存在与唯一性问题在
我们从定的讨论有很大的概率被拉下.

上述 Green 第一公式的证明，对应到课本上定理 3.8.5 (2-D), 3.8.11 (3-D), 3.8.16 (n维)

在 Green 第一公式中，

$$\iint_{\partial D} v \frac{\partial u}{\partial n}\, ds = \iiint_D \underbrace{\nabla v}_{\text{函数}v} \cdot \nabla u\, \underbrace{dV}_{\text{体积元}} + \iiint_D v \Delta u\, \underbrace{dV}_{\text{体积元}}$$

对应地，

$$\iint_{\partial D} u \frac{\partial v}{\partial n}\, ds = \iiint_D \nabla v \cdot \nabla u\, dV + \iiint_D u \Delta v\, dV$$

上式 - 下式

$$\iint_{\partial D} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = \iiint_D (v \Delta u - u \Delta v)\, dV$$

通常被写为

$$\iint_{\partial D} \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial v}{\partial n} \\ u & v \end{vmatrix} ds = \iiint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dV$$

left hand 一阶行列 + ∂D

right hand 二阶行列 + D

~~这就是~~ 这是很完美的. [对应到 thm 3.8.6 (2D) 及 3.8.11 (3D) (第二公式) 及 3.8.17 (n维)]

接着第二公式的应用.

格林第二等式的应用. [不 Sheet 要求]

高老师的工作　2005　1st work　SP
2006　2nd work　BPJ

$$\iiint_D (\vec{\gamma}\Delta\vec{B} - \vec{B}\Delta\vec{\gamma})\,dv = \iint_{\partial D}\left(\vec{\gamma}\frac{\partial\vec{B}}{\partial n} - \vec{B}\frac{\partial\vec{\gamma}}{\partial n}\right)ds$$

$$\begin{vmatrix}\Delta B & \Delta\gamma_x \\ B_x & \gamma_x\end{vmatrix} \qquad \begin{vmatrix}\frac{\partial B_x}{\partial n} & \frac{\partial\gamma_x}{\partial n} \\ B_x & \gamma_x\end{vmatrix}$$

其中: $\begin{bmatrix}\gamma_x\Delta B_x \\ \gamma_y\Delta B_y \\ \gamma_z\Delta B_z\end{bmatrix}_{3\times1} \triangleq \vec{\gamma}\Delta\vec{B}$

$\boxed{\text{分行}\ \text{面微在分片上}}$

推导过程中使用了格林等式的常规操作在 Green 第三公式, 从之前的高斯积分求高斯定律基本处理同时使用了 $\begin{cases}\nabla\times\vec{B} = \lambda\vec{B} \\ \nabla\cdot\vec{B} = 0\end{cases}$

最后得到:

$$C_i\vec{B}_i = \iint_{\partial D}\left(\vec{\gamma}\frac{\partial\vec{B}}{\partial n} - \vec{B}\frac{\partial\vec{\gamma}}{\partial n}\right)ds$$

$\boxed{\text{空间中任意点的磁场}}$

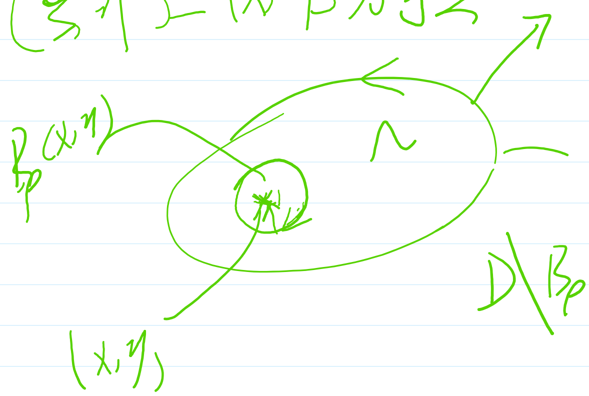
$C_i = \begin{cases}1 & \text{不在 } D \text{ 上} \\ \frac{1}{2} & \text{在 } \partial D \text{ 上}\end{cases}$

这个例子在非线性无力场磁场模型中, 我们会详细的推导.

格林第三公式 (1、2-1)为例

$v(x,y) = \ln\sqrt{(\xi-x)^2+(\eta-y)^2}$ 　令 $r=\sqrt{(\xi-x)^2+(\eta-y)^2}$

$(\xi,\eta)=(x,y)$ 为奇点



$\boxed{\text{人移动时，总在左手边}}$

靠左手边

tips 走到外侧

$D\setminus B_\rho(x,y) = \Omega_\rho$

(单连通，可定向)

G3：已知 u 为可微分函数，Ω 为正则区域，

$C=\partial\Omega$，$r=\sqrt{(\xi-x)^2+(\eta-y)^2}$ 则

$$u(x,y) = \frac{1}{2\pi}\iint_\Omega \ln r\,\Delta u\,d\sigma + \frac{1}{2\pi}\oint_C\left(u\frac{\partial \ln r}{\partial n} - \ln r\frac{\partial u}{\partial n}\right)ds \qquad 3.8.6$$

remark: the right hand of 3.8.6，引入(2-1)，第一项，消去，$\frac{1}{2\pi} < \begin{matrix}\frac{1}{2}\\ 1\end{matrix}$，主要在[illegible]中有 $\begin{cases}\nabla\times B=\mu J\\ \nabla\cdot B=0\end{cases}$ 在计算的过程中的一段[illegible]，π 在计算中可消去。

Proof：(x,y) 是奇点，所以取 $\Omega_\rho = \Omega - B_\rho(x,y)$ 如图。

在 Ω_ρ 中 $v(x,y) = \ln r$

$$= \ln\sqrt{(\xi-x)^2+(\eta-y)^2}$$

所以 $$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2}}\cdot\frac{-2(\xi-x)}{2\sqrt{(\xi-x)^2+(\eta-y)^2}}$$

$$= -\frac{(\xi-x)}{(\xi-x)^2+(\eta-y)^2}$$

$$= -\frac{(\xi - x)}{(\xi-x)^2+(\eta-y)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{-[(\xi-x)^2+(\eta-y)^2]+[(\xi-x)\,2(\xi-x)]}{[(\xi-x)^2+(\eta-y)^2]^2}$$

$$= \frac{(\xi-x)^2-(\eta-y)^2}{[(\xi-x)^2+(\eta-y)^2]^2}$$

同样的2得: $$\frac{\partial^2 v}{\partial y^2} = \frac{(\eta-y)^2-(\xi-x)^2}{[(\xi-x)^2+(\eta-y)^2]^2}$$

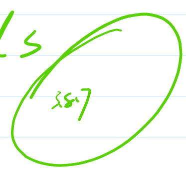
那么: $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2} = 0$$

即 $\Delta V = 0$ i.e. $\Delta \ln r = 0$

在 Ω_ρ 中, 使用 G2, 我们有

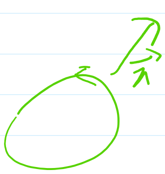
$$\iint_{\Omega_\rho} \begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix} dA = \oint_{\partial\Omega_\rho} \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix} ds$$

$$\iint_{\Omega_\rho} \Delta u\, v\, ds = \oint_{C\cup\partial B_\rho} \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix} ds$$



$$\iint_{\Omega_\rho} \ln r\, \Delta u\, dA = \oint_C \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix} ds - \oint_{\partial B_\rho} \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix} ds$$

$\underbrace{\Omega_\rho,\ C,\ B_\rho}_{\text{boundary}}$



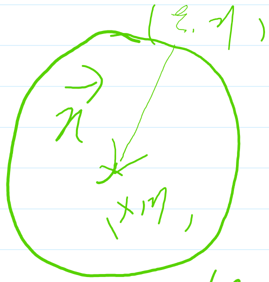
① $$\iint_{\Omega_\rho} \begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix} dA \overset{\Delta v = 0}{=\!=\!=} \iint_{\Omega_\rho} \ln r\, \Delta u\, dA \xrightarrow{\rho \to 0} \iint_{\Omega} \ln r\, \Delta u\, dA$$ ☆

② $$\oint_C \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix} ds = \oint_C \left(\ln r \frac{\partial u}{\partial \vec{n}} - u \frac{\partial \ln r}{\partial \vec{n}}\right) ds$$ ☆☆

$$= -\oint_C \left(u \frac{\partial \ln r}{\partial \vec{n}} - \ln r \frac{\partial u}{\partial \vec{n}}\right) ds$$

$$= -\oint_c \left(u \frac{\partial \ln r}{\partial \vec{n}} - \ln r \frac{\partial u}{\partial \vec{n}}\right) ds \quad ***$$

③ 在 ∂B_ρ 上：



$$\vec{n} = \left(\frac{x-\xi}{r}, \frac{y-\eta}{r}\right)$$

$$(\ln r)' = \frac{1}{r} \quad \frac{}{2\sqrt{\ }}$$

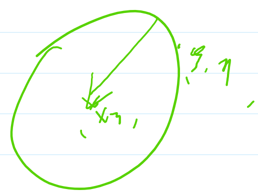
$$\frac{\partial v}{\partial x} = \frac{1}{r} \frac{-2(\xi - x)}{2r}$$

$$= -\frac{\xi - x}{r^2}$$

$$\frac{\partial v}{\partial y} = -\frac{\eta - y}{r^2}$$

$$\frac{\partial v}{\partial \vec{n}} = \frac{(\xi - x)^2 + (\eta - y)^2}{r^2 \, r}$$

$$= \frac{1}{r}$$



$$\frac{\partial u}{\partial x} \boxed{\frac{x-\xi}{r}} + \frac{\partial u}{\partial y} \boxed{\frac{y-\eta}{r}} = \frac{\partial u}{\partial \vec{n}}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial r}{\partial x} = \frac{-(\xi - x)}{r} \qquad \frac{\partial r}{\partial y} = \frac{-(\eta - y)}{r}$$

$$\boxed{\nabla r = \frac{-\vec{r}}{r}} = \vec{n}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x} \frac{\partial (x - \xi)}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial (y - \eta)}{\partial r}$$

$$= \frac{\partial u}{\partial x}\frac{\partial (x-\xi)}{\partial r} + \frac{\partial u}{\partial \eta}\frac{\partial (y-\eta)}{\partial r}$$

$$= \frac{\partial u}{\partial x}\left(-\frac{\partial (\xi - x)}{\partial r}\right) + \frac{\partial u}{\partial \eta}\left(-\frac{\partial (y-\eta)}{\partial r}\right)$$

$$= \nabla u \cdot \left(-\frac{\partial \vec r}{\partial r}\right)$$

$$= \nabla u \cdot \vec n$$

$$\boxed{\begin{array}{l} \text{记 } \vec r = (r\cos\theta,\ r\sin\theta) \\ \frac{\partial \vec r}{\partial r} = (\cos\theta,\ \sin\theta) \\ \frac{\vec r}{r} = (\cos\theta, \sin\theta) \end{array}}$$

故：$\displaystyle\oint_{\partial B_\rho} \begin{vmatrix} \frac{\partial u}{\partial \vec n} & \frac{\partial v}{\partial \vec n} \\ u & v \end{vmatrix} ds = \oint_{\partial B_\rho} \left(\frac{\partial u}{\partial r}\ln r - u\frac{1}{r}\right) ds$

$$= \ln\rho \oint_{\partial B_\rho} \boxed{\frac{\partial u}{\partial \rho}} ds - \frac{1}{\rho}\oint_{\partial B_\rho} u\, ds$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial (x-\xi)}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial (y-\eta)}{\partial r}$$

$$= -\left[\frac{\partial u}{\partial x}\frac{\partial (\xi - x)}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial (y-\eta)}{\partial r}\right]$$

$$= -\left[\frac{\partial u}{\partial x}\frac{\xi - x}{r} + \frac{\partial u}{\partial y}\frac{y-\eta}{r}\right]$$

$$= \nabla u \cdot \vec n \quad ds$$

$$\oint_{\partial B_\rho} \frac{\partial u}{\partial \rho} ds = \oint_{\partial B_\rho} \underline{\nabla u} \cdot \vec n\, ds$$

$$= \iint_{B_\rho} \nabla \cdot (\nabla u)\, dA$$

$$= \iint_{B_\rho} \Delta u\, dA$$

$$\oint_{\partial B_\rho} \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}} \\ u & v \end{vmatrix} ds = \ln\rho \iint_{B_\rho} \Delta u \, dA - \frac{1}{\rho}\oint_{\partial B_\rho} u \, ds$$

因 $\ln\rho \iint_{B_\rho} \Delta u \, dA \le \max|\Delta u| \, \pi\rho^2 \ln\rho \to 0 \quad \rho\to 0^+$

$$\boxed{\lim_{\rho\to 0^+} \rho^2 \ln\rho = \lim_{\rho\to 0^+} \frac{\ln\rho}{\rho^{-2}} \ \boxed{\begin{matrix} -\infty \\ +\infty \end{matrix}} = -\lim \frac{\frac{1}{\rho}}{2\rho^{-3}} = -2\rho^2 \to 0}$$

因為 $\frac{1}{\rho}\oint_{\partial B_\rho} v \, ds = \frac{2\pi}{2\pi\rho}\oint_{\partial B_\rho} u \, ds$

$= \frac{2\pi}{2\pi\rho} \, 2\pi\rho \; u(x,y)$

$= 2\pi \, u(x,y), \quad \rho \to 0$

所以 $\oint_{\partial B_\rho} \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}} \\ u & v \end{vmatrix} ds \to -2\pi u(x,y) \quad \rho\to 0$ (***)

將(*) (**) (***) 代入 3.8.7 中 得到

G_3

ds

在 G_3 中若 $\Delta u = 0$，则

$$u(x,y) = \frac{1}{2\pi}\oint_C \left(u\frac{\partial \ln r}{\partial \vec{n}} - \ln r\frac{\partial u}{\partial \vec{n}}\right)ds$$

约定 3.8.8

利用 3.8.8，可得 3.8.9 平均值定理 2-1)

$\Delta u = 0$，则

$$u(a,b) = \frac{1}{2\pi R}\oint_{C_R} u(x,y)\,ds$$

其中 C_R $(x-a)^2+(y-b)^2=R^2$

3-1) 中的 G_3，令 $V = \frac{1}{r}$

$$= \frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}}$$

可得.

3-1) 中的平均值定理 类似 $\Delta u = 0$

$$u(x,y,z) = \frac{1}{4\pi R^2}\iint_{\partial B_R} u(x,y,z)\,ds$$